For each question following, correct = 7 points, blank = 1.8 points, wrong = 0 point.
Reminder:
(1) Do not urge yourself to finish question(s) in one blow. Relax, this is not a death race.
(2) The difficulty of these questions is built upon the assumption that the writer has less than three contest experiences (regional, states, national…) and knows contents written in the handout of this section on mathnudge.com.

1. Let x and y be such that $x + y = 5$ and $x^3 + y^3 = 2$, give the value of $x^2 + y^2$.
2. The solutions for $\sqrt[3]{9 + x\sqrt{3}} + \sqrt[3]{9 - x\sqrt{3}} = 3$ can be written as $\pm\frac{k\sqrt{w}}{p}$, where k, w, and p are positive integers. Determine the sum $(k + w + p)$.
3. Determine the value of k such that $x^3 + 8x^2 + kx - 2$ is divisible by $(x - 1)$.
4. $2x^3 + 3x^2 - 8x - 12 = (x^2 - a)(bx - c)$. Find the sum $(a + b + c)$.
5. The polynomial $x^3 + 3x^2 - 6x - 12 = 0$ has roots $\{r,\ s,\ t\}$. Meanwhile, polynomial $y^3 + ay^2 + by + c$ has roots $\{r^{-1},\ s^{-1},\ t^{-1}\}$. Determine the sum $(a + b + c)$.
6. Determine the minimum value of $P(x) = -x^2 + 12x - 8$.
7. Factor $x^4 - 4x^3 - 7x^2 + 22x + 24$, a polynomial with only integral roots.
8. Factor $5(y^3 + 8) + 3(y^2 - 4)$.
9. John was solving a quadratic equation with one variable $eq1$, but because he misunderstood the coefficient of the first-degree value as its opposite value, the roots he attained was -4 and 3. What should be the correct solutions for that quadratic equation $eq1$?
10. To make the value of polynomial $20x^2 + 7x + 4$ be 10, what should be the value of x?
11. John and Carter are solving the same quadratic equation. However, while John misunderstood the first-degree term's coefficient and attained roots $-4, -6$; Carter misunderstood the constant term and attained roots $2, 9$. What is the bigger root of this quadratic equation?
12. A continued fraction $a + \frac{b}{a+\frac{b}{a+\frac{b}{a+\frac{b}{\cdots}}}}$ is a fraction with infinite length whose denominator is a quantity q with expression E of a number (a in this case) plus a fraction, with the fraction being expression E. Then, what is the exact value of continued fraction $5 + \frac{2}{5+\frac{2}{5+\frac{2}{5+\frac{2}{\cdots}}}}$?
13. $0.\overline{8} = \frac{c}{q}$, where c and q are coprime. The roots of polynomial $x^2 - qx + c$ are values a, b, where the cubic polynomial $x^3 - ax^2 - bx + c$ has three distinct roots. What is the maximum value of the sum of the three roots of polynomial?
14. $\frac{1}{x^2} + x^2 = 3$, the largest solution for x in this equation can be written in simplified radical form $\frac{k+w\sqrt{p}}{q}$. Determine the sum $(k + w + p + q)$.

Answers Below.

Answer 1

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, and $x+y=5$, $x^3+y^3=2$.
These three equations can be combined into one: $125 = 2 + 15xy$, therefore $xy = \frac{123}{15}$.
$x^2 + y^2 = (x+y)^2 - 2xy = 25 - 2 \times \frac{123}{15} = \boxed{\frac{43}{5}}$.

Answer 2

Substitution: $\sqrt[3]{9 + x\sqrt{3}} = a$, $\sqrt[3]{9 - x\sqrt{3}} = b$. In this case then, $a + b = 3$.
$(a+b)^3 = a^3 + b^3 + 3ab(a+b) = 9 + x\sqrt{3} + 9 - x\sqrt{3} + 9ab = 27$.
This leads to the equation: $ab = 1$, where $ab = \sqrt[3]{(9 + x\sqrt{3})(9 - x\sqrt{3})} = \sqrt[3]{81 - 3x^2}$.
$\left(\sqrt[3]{81 - 3x^2}\right)^3 = 1^3$, and from there we can attain $\frac{80}{3} = x^2$, so $x = \pm\frac{4\sqrt{15}}{3} = \pm\frac{k\sqrt{w}}{p}$.
$k + w + p = 4 + 15 + 3 = \boxed{22}$.

Answer 3

$x^3 + 8x^2 + kx - 2 = 0 = (x-1)(ax^2 + bx + c) = ax^3 + (b-a)x^2 + (c-b)x - c$.
$\begin{cases} a = 1 \\ b - a = 8 \\ c - b = k \\ c = 2 \end{cases}$ this system of equation leads to the value of k: $\boxed{-7}$.

Answer 4

$2x^3 + 3x^2 - 8x - 12 = (x^2 - a)(bx - c) = bx^3 - cx^2 - abx + ac$.
From the above we know that $b = 2$, $c = -3$, and $ab = 8$, $ac = -12$.
$a = 4$, so $a + b + c = 4 + 2 - 3 = \boxed{3}$

Answer 5

Using the Vieta formula, we can attain: $\begin{cases} -(r+s+t) = 3 \\ rs + st + rt = -6 \\ -rst = -12 \end{cases}$.
Using the Vieta formula, we can also attain: $\begin{cases} -(r^{-1} + s^{-1} + t^{-1}) = a \\ r^{-1}s^{-1} + s^{-1}t^{-1} + r^{-1}t^{-1} = b \\ r^{-1}s^{-1}t^{-1} = c \end{cases}$.
Now we can solve for $a,\ b,\ c$.
$a = -\left(\frac{1}{r} + \frac{1}{s} + \frac{1}{t}\right) = -\left(\frac{st}{rst} + \frac{rt}{rst} + \frac{rs}{rst}\right) = -\frac{st+rt+rs}{rst} = -\frac{-6}{12}$, $a = \frac{1}{2}$.
$b = \frac{1}{rs} + \frac{1}{st} + \frac{1}{rt} = \frac{t}{rst} + \frac{r}{rst} + \frac{s}{rst} = \frac{r+s+t}{rst} = -\frac{3}{12}$, $b = -\frac{1}{4}$.
$c = -\frac{1}{rst} = -\frac{1}{12}$.
$a + b + c = \frac{1}{2} - \frac{1}{4} - \frac{1}{12} = \boxed{\frac{1}{6}}$.

Answer 6

$P(x) = -x^2 + 12x - 8$.
Set up a polynomial equation with $P(x)$ and perform some arithmetic given below:
$x^2 - 12x + 8 = 0$, so $x^2 - 12x + 36 = 28 = (x-6)^2$.
$(x-6)^2 - 28 = 0$, and at here if $(x-6)^2$ is at its minimum, which is 0, then the minimum value of polynomial is achieved: $0 - 28 = \boxed{-28}$.

Answer 7

The question hinted that the roots of given polynomial are all integral.
For this polynomial $x^4 - 4x^3 - 7x^2 + 22x + 24$, we can attempt factoring by grouping (which is the seemingly most efficient approach at the moment I see this polynomial).

Because the polynomial can be factored into some form like $(x-k)(x^3 + ax^2 + bx + c)$, which is not its prime factorization, we can first attempt foiling this factored form.
$x^4 - 4x^3 - 7x^2 + 22x + 24 = x^4 + (a-k)x^3 + (b-ka)x^2 + (c-kb)x - kc$, and we know k is an integer. Looking at the Vieta formula, we should notice that a, b, c are all integers as well.
Now, knowing that $kc = -24$, we can set up a case where $c = 24$, $k = -1$, and this case will lead us to the factorization $(x+1)(x^3 - 5x^2 - 2x + 24)$.

Next, we can consider factoring $x^3 - 5x^2 - 2x + 24 = (x-p)(x^2 + lx + m)$.
$x^3 - 5x^2 - 2x + 24 = x^3 + (l-p)x^2 + (m-pl)x - pm$.
Now, knowing that $pm = -24$, we can set up a case where $m = 24$, $p = -1$, and the results of value of l will tell us that this case is wrong, so $m \neq 24$ and $p \neq -1$.[1]
Next, try the case $m = 12$, $p = -2$, and we will attain an appropriate value for $l = -7$, and leads to the factorization $(x+2)(x^2 - 7x + 12)$.

[1] At the above we mentioned "appropriate value for l", and here is an example of how to identify it. If we calculate the case $m = 24$, $p = -1$, then we will get $l = -22$ according to the cubic polynomial's coefficients. However, while $l - p$ is supposed to equal -5, the value $l = -22$ will give a value of $p \neq -1$, which leaves us with two values of p while there should only be one. Because of this phenomenon, we can assure that $m \neq 24$ and $p \neq 1$.
Later in Volume II of upcoming Ars Minor book, we will mention a simplified version of this method: "synthetic division"

From there, $(x^2 - 7x + 12)$ can be factored with cross multiplying into $(x-3)(x-4)$.
The prime factorization of polynomial:
$x^4 - 4x^3 - 7x^2 + 22x + 24 = \boxed{(x+1)(x+2)(x-3)(x-4)}$.

Answer 8

$5(y^3 + 8) + 3(y^2 - 4) = 5(y+2)(y^2 - 2y + 4) + 3(y+2)(y-2)$.
Now we see a common factor of two algebraic expressions: $(y+2)$, so we can do:
$5(y+2)(y^2 - 2y + 4) + 3(y+2)(y-2) = (y+2)(5y^2 - 10y + 20 + 3y - 6)$.
$(y+2)(5y^2 - 10y + 20 + 3y - 6) = \boxed{(y+2)(5y^2 - 7y + 14)}$, and here we completed the prime factorization because we cannot factor $(5y^2 - 7y + 14)$, and this can be verified by using the quadratic formula.

Answer 9

From the description of the problem we can see John's mistaken polynomial was $x^2 + x - 12$, which he misunderstood the first-degree term's coefficient as its opposite value.
That way, the actual polynomial is $x^2 - x - 12 = (x-4)(x+3)$, so the roots are $\boxed{4, -3}$.

Answer 10

The context of problem presents us the equation $20x^2 + 7x + 4 = 10$, $20x^2 + 7x - 6 = 0$.
Factor it whatever way you like, you should get a factorization $(4x+3)(5x-2)$, which shows that $\boxed{x = -\frac{3}{4} \ or \ \frac{2}{5}}$.

Answer 11

We can first call the correct quadratic equation: $ax^2 + bx + c = 0$.
Next, look at John's equation with wrong first-degree coefficient: $x^2 + 10x + 24 = 0$.
From this equation we are certain that $a{:}\,c = 1{:}\,24$.
Then, look at Carter's equation with wrong constant term: $x^2 - 11x + 18 = 0$.
From this equation we are certain that $a{:}\,b = 1{:}\,-11$.
Therefore, the correct quadratic equation would be $x^2 - 11x + 24 = 0$, and its solutions are $3, 8$. The bigger root is $\boxed{8}$.

Answer 12

Define a variable x as $x = 5 + \frac{2}{5+\frac{2}{5+\frac{2}{5+\frac{2}{\cdots}}}}$.
Because this fraction and the denominator of this fraction will continue infinitely, the equation above can also be written as $x = 5 + \frac{2}{x}$, since the denominator of the fraction repeats itself.
Solve the equation $x = 5 + \frac{2}{x}$ for x by multiplying both sides of equation by x, and a quadratic equation appears: $x^2 - 5x - 2 = 0$.
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 1$, $b = -5$, $c = -2$. Then, $x = \boxed{\frac{5 \pm \sqrt{33}}{2} = 5 + \frac{2}{5+\frac{2}{5+\frac{2}{5+\frac{2}{\cdots}}}}}$.

Answer 13

$\frac{c}{q} = 0.\overline{8} = \frac{8}{9}$, $c = 8$ and $q = 9$.
The roots of polynomial $x^2 - qx + c = 0$ then has two roots: 1 and 8.
Polynomial $x^3 - ax^2 - bx + c$ then has two possibilities since a and b are not specified to be which of the two roots.
Possibility 1 gives $x^3 - x^2 - 8x + 8$, which can be factored into $(x^2 - 8)(x - 1)$.
The sum of roots is $\sqrt{8} - \sqrt{8} + 1 = 1$.
Possibility 2 gives $x^3 - 8x^2 - x + 8$, which can be factored into $(x - 8)(x + 1)(x - 1)$.
The sum of roots is $8 - 1 + 1 = 8$.
The maximum of sum is then $\boxed{8}$.

Answer 14

$\frac{1}{x^2} + x^2 = 3$ can be solved by multiplying both sides of equation by x^2.
This renders the equation $x^4 - 3x^2 + 1 = 0$, which, according to the quadratic formula, should give us the result $x^2 = \frac{3 \pm \sqrt{5}}{2}$.
The largest root of above equation should happen in a case where x^2 is its largest possibility, so we should calculate for the case for $x^2 = \frac{3+\sqrt{5}}{2}$, where $x = \sqrt{\frac{3+\sqrt{5}}{2}} = \frac{k + w\sqrt{p}}{q}$, its simplified radical form.
$\left(\frac{k + w\sqrt{p}}{q}\right)^2 = \frac{k^2 + w^2 p}{q^2} + \frac{2kw\sqrt{p}}{q^2} = \frac{3+\sqrt{5}}{2}$, $p = 5$ is indicated in the equation.
This way we can construct two equations based on cross multiplication of fractions:

1. $3q^2 = 2(k^2 + 5w^2)$
2. $4kw = q^2$.

Combining these two equations give: $k^2 - 6kw + 5w^2 = 0 = (k - w)(k - 5w)$.

Depending on the ratio between k and w the equations become:

For $k = w$, $\frac{6w^2}{q^2} = \frac{3}{2}$, and this can be reduced to a ratio $2w = q$.

This means $\frac{k+w\sqrt{p}}{q} = \frac{w+w\sqrt{5}}{2w} = \frac{1+1\sqrt{5}}{2}$, and this answer can be verified since $\left(\frac{1+1\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$.

Meanwhile, for $k = 5w$, $\frac{30w^2}{q^2} = \frac{3}{2}$, and this can be reduced to a ratio $\sqrt{20}w = q$.

This means $\frac{k+w\sqrt{p}}{q} = \frac{5w+w\sqrt{5}}{w\sqrt{20}} = \sqrt{\frac{5}{4}} + \frac{1}{2} = \frac{1+1\sqrt{5}}{2}$.

Any of the two ratios would indicate the answer that $\boxed{(k + w + p + q) = 9}$