Chapter 12-1: System of Equations with 2 Variables

Chapter Guidelines:
- A system of equation is a set of multiple equations.
- Solving a system of equation means attaining the value of variables in those equations and can be achieved in multiple ways.

1: Definition of System of Equations

A system of equations is a set of plural equations.

They are often recorded in a case bracket as how it is shown at right: $\begin{cases} equation\ 1 \\ equation\ 2 \end{cases}$.

Equations of the same system sometimes share variable(s).

For example: in the system $\begin{cases} x = 5 \\ 2x - 1 = 9 \end{cases}$, a variable x is being shared by the equations, and because there is a total of one variable in this system of equation, it is a system of equation with one variable.

There is no limit in how many equations and variables can be included in a system of equations.

2: Methods of Solving Systems of Equations

There are two popular methods for solving systems of equations, and we will introduce both of them below.

1: Method of Substitution

Now we provide a definition for this method of solving systems of equations:

> Method of substitution is a method to solve systems of equations by first producing an identity of a variable, then substituting that variable by its identity in the other equation of the system to solve the value of variable.

In a system of equations with 2 variables, such as the system $\begin{cases} eq1: 2x + y = 4 \\ eq2: 3x - y = 1 \end{cases}$, the shared variables will share a common value. This means the relation between two variables, x and y, remains true when applied to any of the equations in its system.

Method of substitution is built upon this basic concept that allows us to substitute a variable by its identity derived form one of the equations in the system.

In the following example we can observe how method of substitution can be applied to solving a system of equations:

Example: Solve a system of equation $\begin{cases} eq1: 2x + y = 4 \\ eq2: 3x - y = 1 \end{cases}$.
We can modify the equation $eq1$ into $4 - 2x = y$.
A common practice of solving systems of equations is to consider the modified version of an equation a new equation in the system, which would make the system $\begin{cases} eq3: 4 - 2x = y \\ eq2: 3x - y = 1 \end{cases}$.
Now, substitute $eq3$ into $eq2$, and $eq2$ will be modified into an equation $eq4: 3x - (4 - 2x) = 1$.
We will first solve for x in $eq4$, which should give result $x = 1$.
Given that $x = 1$, we can substitute the variable x by the value of this variable in $eq1$ (or any equation of the current system), which will provide $y = 2$.
The solutions to this system are therefore: $\begin{cases} x = 1 \\ y = 2 \end{cases}$.

In the above example we applied method of substitution to a system of linear equations. We may also try applying method of substitution to a system of different types of equations in the below example:

Example: Solve a system of equation $\begin{cases} eq1: x - y = 49 \\ eq2: xy = 102 \end{cases}$.
We can first modify $eq2$ into an equation $eq3: x = \frac{102}{y}$.
Now we can substitute the identity of x attained from $eq3$ into $eq1$, which will provide us an equation $eq4: \frac{102}{y} - y = 49$.
By solving for y in $eq4$, one will also attain the value of x as soon as the mathematician substitutes y by the value of that variable. Therefore, our priority is solving for y, and it can be solved by multiplying both sides of $eq4$ by variable y itself. This action is allowed because $y \neq 0$.
Now $eq4$ will be modified into $eq5: y^2 + 49y - 102 = 0$, and this quadratic equation's solution is that $y = 2\ or - 51$.
If $y = 2$, then $x = 51$; if $y = -51$, $x = -2$. Both sets of solutions are appropriate.
The solutions to this system are therefore: $\begin{cases} x = 51 \\ y = 2 \end{cases}$ or $\begin{cases} x = -2 \\ y = -51 \end{cases}$.

2: Method of Elimination

Now we provide a definition for this method of solving systems of equations:

Method of elimination is a method to solve systems of equations by adding, subtracting equations together to eliminate a variable from the modified equation.

Again, we will demonstrate the usage of method of elimination with an example below:

Example: Solve a system of equation $\begin{cases} eq1: 2x + y = 4 \\ eq2: 3x - y = 1 \end{cases}$.
To use method of elimination, we can produce an equation $eq3 = eq1 + eq2$.
The arithmetic follows: $eq1 + eq2 = 2x + y + 3x - y = 4 + 1$.
That would bring $eq3: 5x = 5$, $x = 1$.
Substitute the value of x into variable x in any equation of the system with the other variable, y, and we will attain the solutions of the system of equations: $\begin{cases} x = 1 \\ y = 2 \end{cases}$.

Practice Questions (No CALC)

Part I. Solve the following systems of equations (If system cites a third variable, solve it as well)

System	Solutions	System	Solutions
(1) $\begin{cases} eq1: 2x + y = 4 \\ eq2: 2x - y = 0 \end{cases}$	$\begin{cases} x = 1 \\ y = 2 \end{cases}$	(2) $\begin{cases} eq1: 3x + 4y = 13 \\ eq2: 6x + y = 19 \end{cases}$	$\begin{cases} x = 3 \\ y = 1 \end{cases}$
(3) $\begin{cases} eq1: x + 4 = 2y \\ eq2: x - 3y = -9 \end{cases}$	$\begin{cases} x = 6 \\ y = 5 \end{cases}$	(4) $\begin{cases} eq1: x + y = 10 \\ eq2: y - x = 4 \end{cases}$	$\begin{cases} x = 3 \\ y = 7 \end{cases}$
(5) $\begin{cases} eq1: 2x - y = 2 \\ eq2: x + y = 22 \end{cases}$	$\begin{cases} x = 8 \\ y = 14 \end{cases}$	(6) $\begin{cases} eq1: xy - 9 = -9 \\ eq2: x + y = 9 \end{cases}$	$\begin{cases} x = 0 \\ y = 9 \end{cases}$
(7) $\begin{cases} eq1: x^2 + y^2 = 13 \\ eq2: 2xy = 120 \end{cases}$	$\begin{cases} x = 12 \\ y = 5 \end{cases}$	(8) $\begin{cases} eq1: x - 2y = 0 \\ eq2: xy - 2^5 = 0 \end{cases}$	$\begin{cases} x = 4 \\ y = 8 \end{cases}$
(9) $\begin{cases} eq1: x + 9 = y \\ eq2: xy = 52 \end{cases}$	$\begin{cases} x = 4 \\ y = 13 \end{cases}$	(10) $\begin{cases} eq1: x^2y = 2156 \\ eq2: xy^2 = 1694 \end{cases}$	$\begin{cases} x = 14 \\ y = 11 \end{cases}$
(11) $\begin{cases} eq1: x = 1.25y \\ eq2: 0.6xy = 108 \end{cases}$	$\begin{cases} x = 15 \\ y = 12 \end{cases}$	(12) $\begin{cases} eq1: y - x = 2 \\ eq2: x^2 - y^2 = -8 \end{cases}$	$\begin{cases} x = 1 \\ y = 3 \end{cases}$
(13) $\begin{cases} eq1: (x + y)^2 = 100 \\ eq2: (y - x)^2 = 16 \end{cases}$	$\begin{cases} x = 7 \\ y = 3 \end{cases}$	(14) $\begin{cases} eq1: (x - y)^2 = 1 \\ eq2: x^2 - y^2 = 17 \end{cases}$	$\begin{cases} x = 9 \\ y = 8 \end{cases}$
(15) $\begin{cases} eq1: xy + 2y = 0 \\ eq2: y = 15 \end{cases}$	$\begin{cases} x = -2 \\ y = 15 \end{cases}$	(16) $\begin{cases} eq1: 5x + 9 = y \\ eq2: y - x = 65 \end{cases}$	$\begin{cases} x = 14 \\ y = 79 \end{cases}$
(17) $\begin{cases} eq1: x(y - x) = 78 \\ eq2: x + y = 32 \\ eq3: y^2 - x^2 = 192 \end{cases}$	$\begin{cases} x = 13 \\ y = 19 \end{cases}$	(18) $\begin{cases} eq1: y - x = 27 \\ eq2: 0.2y - 0.5x = 0 \end{cases}$	$\begin{cases} x = 18 \\ y = 45 \end{cases}$
(19) $\begin{cases} eq1: x = z^2 - 1 \\ eq2: y = z(z + 3) \\ eq3: y - x = 13 \end{cases}$	$\begin{cases} x = 15 \\ y = 28 \\ z = 4 \end{cases}$	(20) $\begin{cases} eq1: 3x - 2y = 0 \\ eq2: y - x = 0.5x \end{cases}$	$\begin{cases} x = 38 \\ y = 57 \end{cases}$
(21) $\begin{cases} eq1: 2x = y \\ eq2: y^2 - x^2 = 192 \end{cases}$	$\begin{cases} x = 8 \\ y = 16 \end{cases}$	(22) $\begin{cases} eq1: x - y = 26 \\ eq2: xy = 407 \end{cases}$	$\begin{cases} x = 37 \\ y = 11 \end{cases}$
(23) $\begin{cases} eq1: x - y = 17 \\ eq2: x^2 - y^2 = 697 \end{cases}$	$\begin{cases} x = 29 \\ y = 12 \end{cases}$	(24) $\begin{cases} eq1: y - \sqrt{x} = 2 \\ eq2: y + x = 158 \end{cases}$	$\begin{cases} x = 144 \\ y = 14 \end{cases}$
(25) $\begin{cases} eq1: xy - 2 = 180 \\ eq2: x = y + 1 \end{cases}$	$\begin{cases} x = 14 \\ y = 13 \end{cases}$	(26) $\begin{cases} eq1: y - x = z^2 \\ eq2: z - 1 = x \\ eq3: 5z - 1 = y \end{cases}$	$\begin{cases} x = 3 \\ y = 19 \\ z = 4 \end{cases}$
(27) $\begin{cases} eq1: y^2 - x^2 = (x - 1)^2 \\ eq2: y - x = 1 \end{cases}$	$\begin{cases} x = 4 \\ y = 5 \end{cases}$	(28) $\begin{cases} eq1: x + y = 15(y - x) \\ eq2: xy = 56 \end{cases}$	$\begin{cases} x = 7 \\ y = 8 \end{cases}$
(29) $\begin{cases} eq1: x - y = 7 \\ eq2: xy^2 = 36 \end{cases}$	$\begin{cases} x = 9 \\ y = 2 \end{cases}$	(30) $\begin{cases} eq1: 3y - 1 = x \\ eq2: xy = 102 \end{cases}$	$\begin{cases} x = 17 \\ y = 6 \end{cases}$