Chapter 11-4: Algebra and Inverse of Matrices

Chapter Guidelines:

- Algebra of Matrices has specific limitations according to how the arithmetic is operated.
- Inverse matrix of a matrix is equivalent to an identity matrix divided by that matrix itself.

1: Algebra of Matrices

We can perform addition and subtraction on matrices that have the same size, which by size we refer to the number of rows and columns a matrix possesses. And so in other words, only matrices with the same numbers of rows and columns can be added together.

The addition of matrices is performed by adding up terms at the same row-column position in matrices and placing them at the same row-column position they were in the sum matrix.
It follows the below pattern:
For an addition of matrices A, B, and C, where: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$,
$a_{ij} + b_{ij} = c_{ij}$; so, for example: $a_{11} + b_{11} = c_{11}$, $a_{12} + b_{12} = c_{12}$, and $a_{21} + b_{21} = c_{21}$.

EX1: $\begin{bmatrix} 8 & 7 \\ 5 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 3 \\ 6 & 8 \end{bmatrix} =?$

Well, according to the above pattern, $\begin{bmatrix} 8 & 7 \\ 5 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 3 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 8+6 & 7+3 \\ 5+6 & 5+8 \end{bmatrix} = \boxed{\begin{bmatrix} 14 & 10 \\ 11 & 13 \end{bmatrix}}$.

Subtraction works the same as addition works for algebra of matrices.
Meanwhile, multiplication of matrices can only happen when the multiplicand is a $m \times n$ matrix and the multiplicator is a $n \times p$ matrix. If that looks a bit too general or confusing, let's look at the rapprochement of that statement below:

The multiplication of matrices: say between a matrix A and a matrix B, can only happen if the number of columns in A is equal to the number of rows in B.

The way multiplication in matrices work follows:

The multiplication of matrices: say between a matrix A and a matrix B, will happen as below pattern demonstrates.

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \times b_{11} + a_{12} \times b_{21} & a_{11} \times b_{12} + a_{12} \times b_{22} \\ a_{21} \times b_{11} + a_{22} \times b_{21} & a_{21} \times b_{12} + a_{22} \times b_{22} \end{bmatrix}$, and the pattern is such that the ij^{th} term of product matrix, $c_{ij} = a_{i1} \times b_{1j} + a_{i2} \times b_{2j} + \cdots + a_{in} \times b_{nj}$.

The denotation n stands for the number of columns in multiplicand as well as the number of rows in multiplicator matrix, as those two values should be equal for the multiplication to successfully carry on.

In other words, $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} \times a_{11} + b_{12} \times a_{21} & b_{11} \times a_{12} + b_{12} \times a_{22} \\ b_{21} \times a_{11} + b_{22} \times a_{21} & b_{21} \times a_{12} + b_{22} \times a_{22} \end{bmatrix}$.

Perhaps this explains the previous rule where the number of columns in multiplicand matrix must equal the number of rows in multiplicator matrix, because of how the multiplication process relies on the equivalence of these two values. If today these two values are different, there would be one value from one of the matrices that finds no other value from the other matrix to multiply with, thus rendering an error in multiplication of matrices.

Meanwhile, it can also be implied that order in matrix multiplication is important, and we will demonstrate with examples:

EX1: $\begin{bmatrix} 8 & 7 \\ 5 & 5 \end{bmatrix} \times \begin{bmatrix} 6 & 3 \\ 6 & 8 \end{bmatrix} = ?$

$\begin{bmatrix} 8 & 7 \\ 5 & 5 \end{bmatrix} \times \begin{bmatrix} 6 & 3 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 8 \times 6 + 7 \times 6 & 8 \times 3 + 7 \times 8 \\ 5 \times 6 + 5 \times 6 & 5 \times 3 + 5 \times 8 \end{bmatrix} = \boxed{\begin{bmatrix} 90 & 80 \\ 60 & 55 \end{bmatrix}}$.

EX2: $\begin{bmatrix} 6 & 3 \\ 6 & 8 \end{bmatrix} \times \begin{bmatrix} 8 & 7 \\ 5 & 5 \end{bmatrix} = ?$

$\begin{bmatrix} 6 & 3 \\ 6 & 8 \end{bmatrix} \times \begin{bmatrix} 8 & 7 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 6 \times 8 + 3 \times 5 & 6 \times 7 + 3 \times 5 \\ 6 \times 8 + 8 \times 5 & 6 \times 7 + 8 \times 5 \end{bmatrix} = \boxed{\begin{bmatrix} 63 & 57 \\ 88 & 82 \end{bmatrix}}$.

Notice that from the two multiplication of matrices above, each with a reversed order of multiplicand and multiplier from each other, the product of the same matrices becomes different depending on the order of multiplication. This means that multiplication of matrices is not commutative, such that $\boxed{A \times B \neq B \times A}$.

Let us summarize our findings about multiplications of matrices into the following rules:

Rule 1: The multiplication between matrices only happen when multiplicand's number of columns equals multiplicator's row number.
Rule 2: Multiplication of an $m \times n$ matrix and $n \times p$ matrix forms a matrix of size $m \times p$.
Rule 3: Multiplication of matrices, where $A \times B = C$, follows the pattern that $\boxed{c_{ij} = a_{i1} \times b_{1j} + a_{i2} \times b_{2j} + \cdots + a_{in} \times b_{nj}}$, where denotation n stands for the number of columns in multiplicand as well as the number of rows in multiplicator matrix.
Rule 4: Multiplication of matrices is not commutative. Order of multiplicand and multiplicator matters.

It may not be very surprising that one can also multiply a matrix by a number.

In a multiplication between a number m and a matrix A:

$$m \times A = mA = m\begin{bmatrix} a_{11} & \cdots & a_{i1} \\ \vdots & \ddots & \vdots \\ a_{1j} & \cdots & a_{ij} \end{bmatrix} = \begin{bmatrix} m \times a_{11} & m \times \cdots & m \times a_{i1} \\ m \times \vdots & m \times \ddots & m \times \vdots \\ m \times a_{1j} & m \times \cdots & m \times a_{ij} \end{bmatrix}.$$

EX1: $-2\begin{bmatrix} 6 & 3 \\ 6 & 8 \end{bmatrix} = ?$

$$-2\begin{bmatrix} 6 & 3 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} -2 \times 6 & -2 \times 3 \\ -2 \times 6 & -2 \times 8 \end{bmatrix} = \boxed{\begin{bmatrix} -12 & -6 \\ -12 & -16 \end{bmatrix}}.$$

EX2: Simplify $\begin{bmatrix} 90 & 80 \\ 60 & 55 \end{bmatrix}$

$$\begin{bmatrix} 90 & 80 \\ 60 & 55 \end{bmatrix} = \begin{bmatrix} 5 \times 18 & 5 \times 16 \\ 5 \times 12 & 5 \times 11 \end{bmatrix} = \boxed{5\begin{bmatrix} 18 & 16 \\ 12 & 11 \end{bmatrix}}.$$

Division of matrices cannot be accomplished, and we will discuss the reasoning of this statement as we discuss inverse matrices.

2: Identity Matrices and Inverse Matrices

To understand what an inverse matrix is, we must first understand an identity matrix.

Identity matrix is a row-echelon form square matrix, where every term in the row right of number 1 is 0. Identity matrix is often denoted as I_n, with n being both the number of columns and rows it has, since identity matrices are square matrices that have the same number of columns and rows (which serves as the reason it is called a square matrix).

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ldots$$

And now we will introduce what an inverse matrix is:

Inverse matrix is a matrix such that for a matrix A, $A \times A^{-1} = I$, where A^{-1} is the inverse matrix of A. This would also mean only square matrices have inverse matrices of themselves.

Inverse matrix has a lot of real-life applications, but perhaps the most relevant one for Algebra II students would be its function in solving a system of equation and detecting the consistency of a system.

If a matrix has an inverse matrix of itself, the matrix is known as an "invertible matrix", since an inverse of it can be founded. Conversely, if a matrix does not have a corresponding inverse matrix, it is a non-invertible matrix.

We can find the inverse matrix of a matrix A through a methodology called Gauss-Jordan method, which is an extension of row operation.

Gauss-Jordan method is a solution of finding an inverse matrix for a matrix A by setting the coefficient matrix of an augmented matrix as matrix A and having an equally sided identity matrix as the argument right to A, or as the part of augmented matrix right of separator.
Next, mathematicians would need to conduct row operations on the augmented matrix until the coefficient matrix A is reduced to an identity matrix of its side. After that, the right argument that was originally an identity matrix would alter into a different matrix, and that matrix would be the inverse of matrix A.

EX1: Find A^{-1} for $A = \begin{bmatrix} 6 & 3 \\ 6 & 8 \end{bmatrix}$.
Step 1: Set up an augmented matrix as the above explanation of Gauss-Jordan method describes:
$\left[\begin{array}{cc|cc} 6 & 3 & 1 & 0 \\ 6 & 8 & 0 & 1 \end{array}\right]$.

Step 2: Conduct row operations on the augmented matrix until the left argument becomes an identity matrix of its size, which would be I_2.

Row Operation	Result
$(\boldsymbol{R2 - R1 \rightarrow R2})$	$\left[\begin{array}{cc\|cc} 6 & 3 & 1 & 0 \\ 0 & 5 & -1 & 1 \end{array}\right]$
$\left(\boldsymbol{\frac{R1}{6} \rightarrow R1}\right)$	$\left[\begin{array}{cc\|cc} 1 & 0.5 & \frac{1}{6} & 0 \\ 0 & 5 & -1 & 1 \end{array}\right]$
$\left(\boldsymbol{R1 - \frac{R2}{10} \rightarrow R1}\right)$	$\left[\begin{array}{cc\|cc} 1 & 0 & \frac{1}{15} & -\frac{1}{10} \\ 0 & 5 & -1 & 1 \end{array}\right]$
$\left(\boldsymbol{\frac{R2}{5} \rightarrow R2}\right)$	$\left[\begin{array}{cc\|cc} 1 & 0 & \frac{1}{15} & -\frac{1}{10} \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5} \end{array}\right]$

Step 3: From above row operations we have attained the inverse matrix as the right-side argument of the augmented matrix: $A^{-1} = \begin{bmatrix} 0.2\overline{6} & -0.1 \\ -0.2 & 0.2 \end{bmatrix}$.

Meanwhile, when dividing a matrix by another, the division of two matrices converts from the expression $\frac{A}{B}$ to $A \times B^{-1}$, while it may also be converted into the multiplication $B \times A^{-1}$.
Due to the non-commutative multiplication process between matrices, divisions between matrices are not applicable.

3: Application of Inverse Matrices

We can use an inverse matrix to solve a system of equations.

We can first develop an expression: $A\begin{bmatrix}x\\y\\z\end{bmatrix} = B$, where A, B are respectively the coefficient matrix and right argument of an augmented matrix originated from a three-variable equation system.

Multiplying both sides of that equation byA^{-1}, which is supposedly a square matrix and the inverse matrix of matrix A, the equation now becomes $\begin{bmatrix}x\\y\\z\end{bmatrix} = A^{-1}B$. We placed the inverse of matrix A as the multiplicand because only this order would make sense for the multiplication equation such that the number of columns in the multiplicand matrix is equivalent to the number of rows in the multiplicator matrix.
That being said, the product $A^{-1}B$ is the solution of this system of equations.

The reason we can use an expression $A\begin{bmatrix}x\\y\\z\end{bmatrix} = B$, where A, B are respectively the coefficient matrix and right argument of an augmented matrix originated from a three-variable equation system can be demonstrated with an example system.
For a system $\begin{cases}a_1x + b_1y + c_1z = d_1\\a_2x + b_2y + c_2z = d_2\\a_3x + b_3y + c_3z = d_3\end{cases}$, where $A = \begin{bmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{bmatrix}$ and $B = \begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}$, the product of matrices A and a matrix $\begin{bmatrix}x\\y\\z\end{bmatrix}$ is $\begin{bmatrix}a_1x + b_1y + c_1z\\a_2x + b_2y + c_2z\\a_3x + b_3y + c_3z\end{bmatrix}$, which equals matrix B.

We can use an example below to demonstrate this rule:

EX1: Solve the system for $\begin{cases}x + 3y = 10\\2x - y = -1\end{cases}$ with inverse matrices.

Step 1: Construct an augmented matrix for this system: $\left[\begin{array}{cc|c}1 & 3 & 10\\2 & -1 & -1\end{array}\right]$.

Step 2: Find the inverse matrix of coefficient matrix. $A = \begin{bmatrix}1 & 3\\2 & -1\end{bmatrix}$, $A^{-1} = \frac{1}{7}\begin{bmatrix}1 & 3\\2 & -1\end{bmatrix}$.

Step 3: $A^{-1}B = \frac{1}{7}\begin{bmatrix}1 & 3\\2 & -1\end{bmatrix}\begin{bmatrix}10\\-1\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{7}\begin{bmatrix}7\\21\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$.

Practice Questions (No CALC)

Part I. Construct the indicated matrices.

Indicated Matrix	Your Construction	Indicated Matrix	Your Construction
$\boldsymbol{I_3}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	I_5	$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$
$\boldsymbol{I_2}$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	I_4	$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

Part II. Conduct indicated algebra between the provided matrices.

Indicated Algebra	**Answer**
(1) $\begin{bmatrix}\mathbf{1}&\mathbf{4}&\mathbf{5}\\\mathbf{2}&\mathbf{7}&\mathbf{4}\end{bmatrix}+\begin{bmatrix}\mathbf{6}&\mathbf{7}&\mathbf{2}\\\mathbf{9}&\mathbf{3}&\mathbf{8}\end{bmatrix}$	$\begin{bmatrix}7&11&7\\11&10&12\end{bmatrix}$
(2) $\begin{bmatrix}\mathbf{12}&\mathbf{43}&\mathbf{6}\\\mathbf{2}&\mathbf{5}&\mathbf{1}\end{bmatrix}+\begin{bmatrix}\mathbf{5}&\mathbf{2}&\mathbf{45}\\\mathbf{12}&\mathbf{87}&\mathbf{63}\end{bmatrix}$	$\begin{bmatrix}17&45&51\\14&92&64\end{bmatrix}$
(3) $\begin{bmatrix}\mathbf{13}&\mathbf{4}&\mathbf{62}\\\mathbf{21}&\mathbf{3}&\mathbf{45}\end{bmatrix}-\begin{bmatrix}\mathbf{9}&\mathbf{7}&\mathbf{34}\\\mathbf{54}&\mathbf{2}&\mathbf{64}\end{bmatrix}$	$\begin{bmatrix}4&-3&28\\-33&1&-19\end{bmatrix}$
(4) $\begin{bmatrix}\mathbf{12}&\mathbf{45}&\mathbf{6}\\\mathbf{24}&\mathbf{64}&\mathbf{86}\end{bmatrix}-\begin{bmatrix}\mathbf{62}&\mathbf{53}&\mathbf{12}\\\mathbf{12}&\mathbf{45}&\mathbf{23}\end{bmatrix}$	$\begin{bmatrix}-50&-8&-6\\12&19&63\end{bmatrix}$
(5) $\begin{bmatrix}\mathbf{1}&\mathbf{5}&\mathbf{7}\\\mathbf{3}&\mathbf{6}&\mathbf{8}\end{bmatrix}\times\begin{bmatrix}\mathbf{2}&\mathbf{6}\\\mathbf{3}&\mathbf{7}\\\mathbf{5}&\mathbf{9}\end{bmatrix}$	$\begin{bmatrix}52&104\\64&132\end{bmatrix}$
(6) $\begin{bmatrix}\mathbf{12}&\mathbf{5}&\mathbf{4}\\\mathbf{2}&\mathbf{7}&\mathbf{-10}\end{bmatrix}\times\begin{bmatrix}\mathbf{2}&\mathbf{5}\\\mathbf{-12}&\mathbf{2}\\\mathbf{2}&\mathbf{5}\end{bmatrix}$	$\begin{bmatrix}-28&90\\-100&-26\end{bmatrix}$
(7) $\begin{bmatrix}\mathbf{42}&\mathbf{1}&\mathbf{4}\\\mathbf{53}&\mathbf{6}&\mathbf{2}\end{bmatrix}\times\begin{bmatrix}\mathbf{4}&\mathbf{-4}\\\mathbf{-24}&\mathbf{3}\\\mathbf{2}&\mathbf{-15}\end{bmatrix}$	$\begin{bmatrix}152&-225\\72&-224\end{bmatrix}$
(8) $\begin{bmatrix}\mathbf{23}&\mathbf{4}&\mathbf{5}\\\mathbf{6}&\mathbf{8}&\mathbf{0}\end{bmatrix}\times\begin{bmatrix}\mathbf{2}&\mathbf{5}\\\mathbf{0}&\mathbf{7}\\\mathbf{0}&\mathbf{9}\end{bmatrix}$	$\begin{bmatrix}46&188\\12&86\end{bmatrix}$
(9) $\begin{bmatrix}\boldsymbol{x}&\boldsymbol{y}&\boldsymbol{z}\end{bmatrix}-\begin{bmatrix}\boldsymbol{x}&\boldsymbol{y}&\boldsymbol{z}\end{bmatrix}$	$\begin{bmatrix}0&0&0\end{bmatrix}$
(10) $\begin{bmatrix}\mathbf{2}&\mathbf{5}\\\boldsymbol{d}&\boldsymbol{y}\end{bmatrix}+\begin{bmatrix}\mathbf{2}&\mathbf{4}\\\mathbf{2}\boldsymbol{d}&-\boldsymbol{y}\end{bmatrix}$	$\begin{bmatrix}4&20\\3d&0\end{bmatrix}$
(11) $\begin{bmatrix}\mathbf{2}&\mathbf{3}&\mathbf{4}\\\mathbf{8}&\mathbf{1}&\mathbf{5}\\\mathbf{9}&\mathbf{7}&\mathbf{6}\end{bmatrix}\times\mathbf{3}\begin{bmatrix}\mathbf{1}&\mathbf{2}&\mathbf{3}\\\mathbf{2}&\mathbf{2}&\mathbf{2}\\\mathbf{2}&\mathbf{3}&\mathbf{2}\end{bmatrix}$	$\begin{bmatrix}48&66&60\\60&99&108\\105&150&159\end{bmatrix}$
(12) $\mathbf{4}\begin{bmatrix}\mathbf{1}&\mathbf{2}&\mathbf{4}\\\mathbf{2}&\mathbf{3}&\mathbf{5}\\\mathbf{0}&\mathbf{9}&\mathbf{1}\end{bmatrix}\times\begin{bmatrix}\mathbf{1}&\mathbf{2}&\mathbf{2}\\\mathbf{6}&\mathbf{2}&\mathbf{5}\\\mathbf{14}&\mathbf{3}&\mathbf{4}\end{bmatrix}$	$\begin{bmatrix}276&72&112\\360&100&156\\272&84&196\end{bmatrix}$

(13) $\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \\ \mathbf{5} & \mathbf{6} \\ \mathbf{7} & \mathbf{8} \end{bmatrix} \times \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{bmatrix}$	$\begin{bmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \\ 39 & 52 & 69 \end{bmatrix}$
(14) $\begin{bmatrix} \mathbf{5} & \mathbf{2} \\ \mathbf{3} & \mathbf{7} \\ \mathbf{1} & \mathbf{5} \end{bmatrix} \times \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{1} \end{bmatrix}$	$\begin{bmatrix} 9 & 16 & 23 & 30 & 12 \\ 17 & 27 & 37 & 47 & 13 \\ 11 & 17 & 23 & 29 & 7 \end{bmatrix}$
(15) $\begin{bmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{6} & \mathbf{4} \end{bmatrix} + \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{6} \\ \mathbf{8} & \mathbf{9} & \mathbf{3} \end{bmatrix} \times \begin{bmatrix} \mathbf{2} & \mathbf{15} \\ \mathbf{1} & \mathbf{8} \\ \mathbf{25} & \mathbf{12} \end{bmatrix}$	$\begin{bmatrix} 155 & 107 \\ 106 & 232 \end{bmatrix}$
(16) $\begin{bmatrix} \mathbf{1} \\ \mathbf{24} \\ \mathbf{5} \end{bmatrix} \times \mathbf{12}\begin{bmatrix} \mathbf{6} & \mathbf{2} & \mathbf{2} \end{bmatrix} + [\mathbf{4}]$	$[772]$
(17) $\begin{bmatrix} \mathbf{2} & \mathbf{5} \\ \mathbf{2} & \mathbf{3} \\ \mathbf{10} & \mathbf{18} \end{bmatrix} \times \frac{1}{5}\begin{bmatrix} \mathbf{3} \\ \mathbf{2} \end{bmatrix} + \begin{bmatrix} \mathbf{23} \\ \mathbf{32} \\ \mathbf{33} \end{bmatrix}$	$\begin{bmatrix} 26.2 \\ 34.4 \\ 46.2 \end{bmatrix}$
(18) $\frac{2}{3}\begin{bmatrix} \mathbf{12} & \mathbf{42} & \mathbf{6} \\ \mathbf{27} & \mathbf{15} & \mathbf{24} \\ \mathbf{21} & \mathbf{36} & \mathbf{9} \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \mathbf{12} & \mathbf{28} \\ \mathbf{24} & \mathbf{52} \\ \mathbf{56} & \mathbf{20} \end{bmatrix} \times \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} & \mathbf{0} \end{bmatrix}$	$\begin{bmatrix} 25 & 41 & 4 \\ 50 & 35 & 16 \\ 38 & 57 & 6 \end{bmatrix}$
(19) $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{4} & \mathbf{1} \end{bmatrix} \times \mathbf{0.\overline{8}}\begin{bmatrix} \mathbf{18} \\ \mathbf{9} \\ \mathbf{36} \end{bmatrix} - \begin{bmatrix} \mathbf{24} \\ \mathbf{24} \\ \mathbf{23} \end{bmatrix}$	$\begin{bmatrix} -8 \\ 16 \\ 73 \end{bmatrix}$
(20) $\begin{bmatrix} \mathbf{11} & \mathbf{29} & \mathbf{12} & \mathbf{20} \\ \mathbf{14} & \mathbf{2} & \mathbf{6} & \mathbf{5} \\ \mathbf{8} & \mathbf{7} & \mathbf{6} & \mathbf{3} \end{bmatrix} - \begin{bmatrix} \mathbf{10} & \mathbf{16} & \mathbf{2} \\ \mathbf{4} & \mathbf{2} & \mathbf{7} \\ \mathbf{12} & \mathbf{3} & \mathbf{5} \end{bmatrix} \times \begin{bmatrix} \mathbf{8} & \mathbf{9} & \mathbf{2} & \mathbf{6} \\ \mathbf{6} & \mathbf{3} & \mathbf{7} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} & \mathbf{7} & \mathbf{8} \end{bmatrix}$	$\begin{bmatrix} -167 & -115 & -134 & -88 \\ -37 & -61 & -65 & -79 \\ -111 & -125 & -74 & -115 \end{bmatrix}$

Part III. Find the inverse matrix for the following provided square matrices. If the matrix is not invertible, note “not invertible”.

Matrix	Inverse Matrix	Matrix	Inverse Matrix
(1) $\begin{bmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{1} & \mathbf{3} \end{bmatrix}$	Not invertible	(2) $\begin{bmatrix} 2 & 4 \\ 2 & 5 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 5 & -4 \\ -2 & 2 \end{bmatrix}$
(3) $\begin{bmatrix} \mathbf{7} & \mathbf{9} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}$	$\frac{1}{5}\begin{bmatrix} 2 & -9 \\ -1 & 7 \end{bmatrix}$	(4) $\begin{bmatrix} 4 & 7 \\ 1 & 5 \end{bmatrix}$	$\frac{1}{13}\begin{bmatrix} 5 & -7 \\ -1 & 4 \end{bmatrix}$
(5) $\begin{bmatrix} \mathbf{7} & \mathbf{1} \\ \mathbf{14} & \mathbf{2} \end{bmatrix}$	Not invertible	(6) $\begin{bmatrix} 5 & 5 \\ 5 & 6 \end{bmatrix}$	$\frac{1}{5}\begin{bmatrix} 6 & -5 \\ -5 & 5 \end{bmatrix}$
(7) $\begin{bmatrix} \mathbf{2} & \mathbf{4} \\ \mathbf{2} & \mathbf{24} \end{bmatrix}$	$\frac{1}{20}\begin{bmatrix} 12 & -2 \\ -1 & 1 \end{bmatrix}$	(8) $\begin{bmatrix} 12 & 5 \\ 6 & 22 \end{bmatrix}$	$\frac{1}{234}\begin{bmatrix} 22 & -5 \\ -6 & 12 \end{bmatrix}$
(9) $\begin{bmatrix} \mathbf{7} & \mathbf{12} \\ \mathbf{1} & \mathbf{-7} \end{bmatrix}$	$\frac{1}{61}\begin{bmatrix} 7 & 12 \\ 1 & -7 \end{bmatrix}$	(10) $\begin{bmatrix} -24 & 5 \\ 7 & 8 \end{bmatrix}$	$\frac{1}{227}\begin{bmatrix} -8 & 5 \\ 7 & 24 \end{bmatrix}$

(11) $\begin{bmatrix} \mathbf{1} & \mathbf{4} & \mathbf{6} \\ \mathbf{2} & \mathbf{-3} & \mathbf{7} \\ \mathbf{8} & \mathbf{5} & \mathbf{2} \end{bmatrix}$	$\frac{1}{371}\begin{bmatrix} -41 & 22 & 46 \\ 52 & -46 & 5 \\ 34 & 27 & -11 \end{bmatrix}$	(12) $\begin{bmatrix} 6 & -3 & 6 \\ 2 & -1 & 2 \\ -2 & 1 & -2 \end{bmatrix}$	Not invertible
(13) $\begin{bmatrix} \mathbf{4} & \mathbf{2} & \mathbf{7} \\ \mathbf{1} & \mathbf{0} & \mathbf{4} \\ \mathbf{-5} & \mathbf{4} & \mathbf{5} \end{bmatrix}$	$\frac{1}{86}\begin{bmatrix} 16 & -18 & -8 \\ 25 & -55 & 9 \\ -4 & 26 & 2 \end{bmatrix}$	(14) $\begin{bmatrix} 4 & 0 & 2 \\ 2 & 0 & 2 \\ 8 & 2 & 3 \end{bmatrix}$	$\frac{1}{4}\begin{bmatrix} 2 & -2 & 0 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}$
(15) $\begin{bmatrix} \mathbf{7} & \mathbf{9} & \mathbf{0} \\ \mathbf{2} & \mathbf{6} & \mathbf{3} \\ \mathbf{2} & \mathbf{7} & \mathbf{9} \end{bmatrix}$	$\frac{1}{123}\begin{bmatrix} 33 & -81 & 27 \\ -12 & 63 & -21 \\ 2 & -31 & 24 \end{bmatrix}$	(16) $\begin{bmatrix} 2 & 4 & 8 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix}$	Not invertible
(17) $\begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{5} & \mathbf{1} \\ \mathbf{4} & \mathbf{5} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{5} & \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{1} \end{bmatrix}$	$\frac{1}{44}\begin{bmatrix} -48 & 9 & -17 & 46 \\ 4 & -9 & 17 & -2 \\ 20 & -1 & -3 & -10 \\ 28 & 14 & -2 & -36 \end{bmatrix}$	(18) $\begin{bmatrix} 1 & 4 & 6 & 2 \\ 3 & 4 & 7 & 5 \\ 2 & 1 & 5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	Not invertible
(19) $\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{6} & \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{4} & \mathbf{7} & \mathbf{2} & \mathbf{2} \\ \mathbf{2} & \mathbf{5} & \mathbf{8} & \mathbf{3} \end{bmatrix}$	Not invertible	(20) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 5 & 0 & 3 \end{bmatrix}$	Not invertible

Part IV. Solve the following systems using (a) row operation and (b) inverse matrices, and check if the two methods provide the same solution (as they should). If the coefficient matrix is not invertible, only solve the system with row operation and report the result.

System	Solution
(1) $\begin{cases} \mathbf{2x + 1 = y} \\ \mathbf{2 - y = -x} \end{cases}$	$\begin{cases} x = 1 \\ y = 3 \end{cases}$
(2) $\begin{cases} \mathbf{2x + y = 2} \\ \mathbf{3x - 6y = -12} \end{cases}$	$\begin{cases} x = 0 \\ y = 2 \end{cases}$
(3) $\begin{cases} \mathbf{2y + 3 = x} \\ \mathbf{2x - 4y = 6} \end{cases}$	No Solution
(4) $\begin{cases} \mathbf{3x - 9 = y} \\ \mathbf{x + y = 11} \end{cases}$	$\begin{cases} x = 5 \\ y = 6 \end{cases}$
(5) $\begin{cases} \mathbf{x + 2y = 7} \\ \mathbf{x - y = 4} \end{cases}$	$\begin{cases} x = 5 \\ y = 1 \end{cases}$
(6) $\begin{cases} \mathbf{3x = y - x} \\ \mathbf{24x = 36} \end{cases}$	$\begin{cases} x = 1.5 \\ y = 6 \end{cases}$
(7) $\begin{cases} \mathbf{x - y = 0} \\ \mathbf{x + y = 2} \end{cases}$	$\begin{cases} x = 1 \\ y = 1 \end{cases}$
(8) $\begin{cases} \mathbf{2x + 2y = 24} \\ \mathbf{x - y = 1} \end{cases}$	$\begin{cases} x = 6.5 \\ y = 5.5 \end{cases}$
(9) $\begin{cases} \mathbf{3x - y + z = 31} \\ \mathbf{x + y - z = 13} \\ \mathbf{x - y - z = -5} \end{cases}$	$\begin{cases} x = 11 \\ y = 9 \\ z = 7 \end{cases}$

(10) $\begin{cases} 2x - y + z = 9 \\ x - 3y + 5z = -2 \\ 2x + y - 2z = 16 \end{cases}$	$\begin{cases} x = 7 \\ y = 8 \\ z = 3 \end{cases}$
(11) $\begin{cases} x - 2y + z = 2 \\ x - y + z = 8 \\ -x - 3y + z = -28 \end{cases}$	$\begin{cases} x = 12 \\ y = 6 \\ z = 2 \end{cases}$
(12) $\begin{cases} x - y + z = 19 \\ x - 8y + z = 12 \\ 2x + 2y - 0.5z = 12 \end{cases}$	$\begin{cases} x = 8 \\ y = 1 \\ z = 12 \end{cases}$
(13) $\begin{cases} 2x - 2y + z = 18 \\ 3x - 2y + z = 26 \\ 5x + 6y - 7z = 10 \end{cases}$	$\begin{cases} x = 8 \\ y = 2 \\ z = 6 \end{cases}$
(14) $\begin{cases} w + x - (y + z) = 4 \\ w - (x - y) - 3z = 0 \\ w + x - y - 2z = 4 \\ 2w - 3x + y - z = 0 \end{cases}$	$\begin{cases} w = 16 \\ x = 15 \\ y = 20 \\ z = 7 \end{cases}$
(15) $\begin{cases} w - (x - y) - z = -1 \\ w + x - y - 7.5z = 1 \\ w - 2x + y + z = -10 \\ 2w - x - y - z = 10 \end{cases}$	$\begin{cases} w = 17 \\ x = 17 \\ y = 3 \\ z = 4 \end{cases}$