Chapter 11-5: Determinant and Its Properties

Chapter Guidelines:

- Determinant is a value that indicates many properties of a matrix.
- Because so, determinants have a lot of applications to use.

Before we start, I hope we can discuss about the element notations we have been using so far, known as a_{mn}, which we also used when expressing cofactors and minors.
The inconvenience in this notation is that once m or n exceeds 10, the recording of notation becomes difficult to comprehend.
For example, the element of a matrix at 10^{th} row and 11^{th} column would be expressed as a_{1011} according to our current denotation rules, but it will be hard to recognize whether a_{1011} actually means the element of at 10^{th} row and 11^{th} column or at 101^{th} row and 1^{st} column.
To solve that problem, mathematicians often use the notation $a_{m,n}$ rather than a_{mn}, where the column between m and n would help distinguish the values of m and n.

Therefore, I will start using the notation $a_{m,n}$ from now on, since it is a clearer notation to use.

1: Definition and Calculation of Determinant

Determinant is a quantity that is calculated by adding and subtracting products of specific elements in square matrices, and relates to several properties of a matrix, and qualities of applications of matrices we will see in Precalculus.
For example, the determinant of a non-invertible matrix is always 0, while the determinant of an invertible matrix is never zero.
Determinants can also be used to solve systems of equations.

In common usage, the determinant of a matrix A is either $\det(A)$ or $|A|$.
Meanwhile, since mathematicians can only calculate the determinant of square matrices, we will introduce the calculation of determinants based on the size of square matrices, first introducing the determinant of 2×2 square matrix:

> The determinant of a 2×2 square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be computed as $\boxed{ad - bc}$.

Based on the above definition, we can look into the determinant of a 3×3 square matrix:

> The determinant of matrices larger than or equal to 3×3 is equivalent to the sum products of top-row elements and their cofactors.

To understand the above statements, we need to know these vocabularies first: cofactors and minors. We will discuss about minors first.

A minor is the determinant of a matrix resulting by excluding the column and row that a specific element $a_{i,j}$ is included in, often denoted as $M_{i,j}$.

I am to demonstrate the visualization and calculation of minors of a_{11}, a_{12}, and a_{13} for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix}$ in below section:

In the below visualization, the bolded, green-texted element is the element this minor belongs to, which we will call a_{ij}.

Matrix of $M_{1,1}$	Matrix of $M_{1,2}$	Matrix of $M_{1,3}$
$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & 3 & 4 \\ \mathbf{3} & 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 1 \end{bmatrix}$	$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 2 & \mathbf{3} & 4 \\ 3 & \mathbf{4} & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$	$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 2 & 3 & \mathbf{4} \\ 3 & 4 & \mathbf{1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

Elements with green highlights are elements of same row or column as $a_{i,j}$ does. Meanwhile, the right-side of equation gives the matrix of $a_{i,j}$'s minor that we calculate determinant of.
The minor for element $a_{1,1}$ in this matrix will be $\det\left(\begin{bmatrix} 3 & 4 \\ 4 & 1 \end{bmatrix}\right) = 3 - 16 = -13$.
Meanwhile the minor for element $a_{1,2}$ is $\det\left(\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}\right) = 2 - 12 = -10$.
The minor for element $a_{1,3}$ is $\det\left(\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}\right) = 8 - 9 = -1$.

Now we will discuss cofactors.
Here is a definition provided below:

A cofactor is the product of an element $a_{i,j}$'s place sign and its minor, denoted as $C_{i,j}$

We will discuss the definition and formulation of "place signs", a term we mentioned in cofactor's definition with, again, the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix}$:

Place signs are positive/negative signs distributed at specific positions in the matrix array that follows a similar pattern to a checkerboard:
Adjacent to every place sign is the opposite of each other, just like adjacent to all black checks in checkerboard are only white checks.

The place sign for a 3×3 square matrix is: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$; that for a 4×4 matrix is: $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$.

If we observe closer, we will find the positive place signs only appear at elements $a_{i,j}$ where $i + j$ is even, while negative place signs appear only at elements $a_{i,j}$ where $i + j$ is odd.
Therefore, the place sign of an element in square matrix can be summarized into the following formula: $Place\ sign\ of\ a_{i,j} = (-1)^{i+j}$.

The cofactor of an element a_{ij} is summarized by a formula $C_{i,j} = (-1)^{i+j} M_{i,j}$.
$C_{1,1} = (-1)^{1+1} \times (3 - 16) = -13$, $C_{1,2} = (-1)^{1+2} \times (2 - 12) = 10$, $C_{1,3} = (-1)^{1+3} \times (8 - 9) = -1$.

Finally, having observed and understood terminologies mentioned in the definition of determinants for $\geq 3 \times 3$ square matrices, we are to revisit the definition below.

"The determinant of matrices larger than or equal to 3×3 is equivalent to the sum products of top-row elements and their cofactors, which are products of top-row elements' place signs and their minors."

This statement indicates the formula: $\det(n \times n\ sized\ matrix\ A) = a_{1,1} \times C_{1,1} + a_{1,2} \times C_{1,2} + \cdots + a_{1,n} \times C_{1,n}$, while $C_{ij} = (-1)^{i+j} M_{i,j}$, while $M_{i,j}$ is the determinant of matrix A excluding the i^{th} row and j^{th} column.

So, for matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix}$:
$\det(A) = a_{1,1} \times C_{1,1} + a_{1,2} \times C_{1,2} + a_{1,3} \times C_{1,3}$, where $C_{i,j} = (-1)^{i+j} M_{i,j}$.
For $\boxed{minors}$:
$M_{1,1} = (3 - 16) = -13$, $M_{1,2} = (2 - 12) = -10$, $M_{1,3} = (8 - 9) = -1$.
For $\boxed{cofactors}$:
$C_{1,1} = (-1)^{1+1} \times (3 - 16) = -13$, $C_{1,2} = (-1)^{1+2} \times (2 - 12) = 10$, $C_{1,3} = (-1)^{1+3} \times (8 - 9) = -1$.
Last but not least, for $\boxed{determinant}$:
$a_{1,1} \times C_{1,1} + a_{1,2} \times C_{1,2} + a_{1,3} \times C_{1,3} = 1 \times -13 + 2 \times 10 + 3 \times -1 = 4$, so $\det(A) = 4$.

Based on the definition of determinants for matrices, we can develop a formula for 3×3 square matrices' determinants:
For a matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$.

With above knowledges, we can finally develop a definition of determinant and its calculations:

Determinant is a quantity that only square matrices possess and is calculated as the sum of products of top row elements and their respective cofactors, while determinant for a 1×1 matrix is its element's value.

Meanwhile, there are some significant properties of determinants:

1. $\det(I) = 1$
2. If two rows were exchanged in a matrix, its determinant becomes its opposite value.
3. If one row was multiplied by a number in matrix, its determinant is multiplied by that number as well. The addition and multiplication in matrices reflect to its determinants linearly.

EX: $x \times \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \det\left(\begin{bmatrix} ax & bx \\ c & d \end{bmatrix}\right)$, $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + \det\left(\begin{bmatrix} a & e \\ c & f \end{bmatrix}\right) = \det\left(\begin{bmatrix} a & b+e \\ c & d+f \end{bmatrix}\right)$.

4. Addition and subtractions of rows within a matrix does not affect its determinant.
5. $\det(AB) = \det(A) \times \det(B)$.

2: Application of Determinants: Finding Inverse Matrix

We can also use the determinant of a matrix to find its inverse matrix by operating the following steps. In the following section, we will define each step and provide an example simultaneously in description columns with the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix}$.

Now we will demonstrate Step 1 of finding inverse matrices: Replacement by minors.

This step demands the mathematician (you, in this case) to replace each element of the matrix by its minor.

Operations: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -13 & -10 & -1 \\ -10 & 0 & -2 \\ -1 & -2 & -1 \end{bmatrix}$.

Step 2: Replacement by Cofactors

This step demands the mathematician (you again) to replace each element of the matrix after step 1 by its cofactor.

Operations: $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$, $\begin{bmatrix} -13 & -10 & -1 \\ -10 & 0 & -2 \\ -1 & -2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -13 & 10 & -1 \\ 10 & 0 & 2 \\ -1 & 2 & -1 \end{bmatrix}$.

Step 3: Adjugate Matrix

Now, imagine that the diagonal running up-left to down-right is an axis of reflection and exchange elements that are at the corresponding location across halves of the matrix divided by the aforementioned diagonal. This step produces the adjugate matrix, or adjoint of original matrix.

Operations: $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -13 & 10 & -1 \\ 10 & 0 & 2 \\ -1 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -13 & 10 & -1 \\ 10 & 0 & 2 \\ -1 & 2 & -1 \end{bmatrix}.$

Step 4: Multiply the entire matrix by the reciprocal of original matrix's determinant

This operation is what it literally says.
Operations:

$\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \rightarrow \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -13 & 10 & -1 \\ 10 & 0 & 2 \\ -1 & 2 & -1 \end{bmatrix} \rightarrow \frac{1}{4}\begin{bmatrix} -13 & 10 & -1 \\ 10 & 0 & 2 \\ -1 & 2 & -1 \end{bmatrix}.$

And I would add a Step 5: Verification, using the specific property $A^{-1}A = AA^{-1} = I$.

Operations:

For the 2×2 matrix: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

For the 3×3 matrix: $\frac{1}{4}\begin{bmatrix} -13 & 10 & -1 \\ 10 & 0 & 2 \\ -1 & 2 & -1 \end{bmatrix}.$

3: Application of Determinants: Cramer's Rule

We have previously mentioned that we can use determinants to solve a system of equations, and the operation that enables this solution is called "Cramer's Rule".

Cramer's Rule relates to the utilization of augmented matrix and expresses the solutions of a system of linear equations in terms of determinants.

A system $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$ has an augmented matrix $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$.

Cramer Rule defined some denotations to represent a system's solutions. For example, notation D stands for the determinant of coefficient matrix. Meanwhile, notation D_m, with m being a variable, is the determinant of the coefficient matrix with the column representing m's coefficients replaced by the right argument of augmented matrix, which is the row of ds.

Therefore, $D_x = \det\left(\begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix}\right)$, and $D_y = \det\left(\begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix}\right)$.

Cramer's Rule states that the solution for variable $m = \frac{D_m}{D}$. Thus, the solution to the above system would be $(\frac{D_x}{D}, \frac{D_y}{D}, \frac{D_z}{D})$.

The derivation of Cramer's Rule follows, but I would like to add a few notes onto the following derivation.
The derivation below is mostly based on multiple Internet sources, which I later translate and summarize into the following, so some actions may be lost in translation. Meanwhile, it is extremely reasonable to get confused by it, as the following contents are preserved for those who are interested in finding out more about how the learned knowledge work. It is fine to be confused. Process it through a couple times if needed.

We will first construct a basic understanding about getting solutions of systems and establish some basic symbols:

"Cramer's Rule relates to the utilization of augmented matrix and expresses the solutions of a system of linear equations in terms of determinants."

This means we can reuse something we learned in the last chapter: write an entire linear system as the expression $Ax = B$, where B is the right argument of augmented matrix for the system and A is the coefficient matrix, and x is the single-column matrix that represents variables of system.

So say that we have k variables to solve for, it would make A an $k \times k$ square matrix, and both X and B are $k \times 1$ matrices. To figure out X, we have to attain the product $A^{-1}B$, and not the other arrangement where B is the multiplicand because it would not cause a multiplication between matrices A^{-1} and B.

In this derivation, let's call our variables $x_1, x_2, x_3 \dots x_k$ instead of calling them x, y, z ..., and define matrix X_n as an altered version of I_k where column n is replaced by a column of variables and $n \leq k$.

For example, for $k = 3$, X_2 will be $\begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix}$, and X_3 will be $\begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & x_3 \end{bmatrix}$.

Meanwhile, since in Cramer's rule a variable $m = \frac{D_m}{D}$, we will define a matrix M_m that has the determinant D_m by being the altered version of A whose m^{th} column is replaced by B's only column.

Now we will go on with the complicated parts:

If we multiply 2 matrices in the arrangement $A \times X_n$, where $A = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}$ and X_n is an altered version of I_k whose n^{th} column becomes X, the single-column matrix of all variables, then the product AX_n would have every column other than column n be the same as its corresponding column with matrix A, while its n^{th} column becomes exactly what the also single-columned matrix B is. This means $AX_n = M_n$.
The equation $AX_n = M_n$ tells us $|A||X_n| = |M_n|$, based on the properties of determinant.

So what is the determinant of matrix X_n?

The matrix X_n is an altered version of identity matrix, such that the n^{th} column is the contents of matrix X and the n^{th} row consists of 0s and x_n at the n^{th} column. This means that the determinant of matrix X_n, which is calculated through sums of cofactors and elements of specific row, would only be $(-1)^{row+column} x_n \times \mathrm{M}_{\mathrm{n,n}}$ where $n \in \mathbb{Z}$.
The sum of row number and column number is $2n$, which must be an even integer since $n \in \mathbb{Z}$.
This makes $(-1)^{row+column} = (-1)^{2n} = 1$. Meanwhile, $M_{n,n}$ that is in fact x_n per se, would be equivalent to the determinant of identity matrix. This can be demonstrated by writing out the matrix X_n, then crossing out the column and row $a_{n,n}$ locates at.
So in conclusion, $|X_n| = x_n$.

To clarify on the minor of element at n^{th} row and n^{th} column of X_n, I will construct the matrix of X_n below.

$$X_n = \begin{bmatrix} 1 & \cdots & x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & x_n & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & x_k & \cdots & 1 \end{bmatrix}$$

Once the highlighted row and column are eliminated to find minor, we can notice that finding minor is equivalent to finding the determinant of an identity matrix.
Then, since $\det(I) = 1$, $M_{n,n} = 1$.

So in conclusion:

$AX_n = M_n$, which means $\det(A) \times \det(X_n) = \det(M_n)$.
This can be derived into $\det(X_n) = \frac{\det(M_n)}{\det(A)}$, and since $\det(X_n) = x_n$, $x_n = \frac{\det(M_n)}{\det(A)} = \frac{D_n}{D}$.

And here ends the derivation of Cramer's Rule, a method of solving system that is often seen as ineffective.
Below is an example of Cramer's Rule.

EX: Solve the system: $\begin{cases} x - y + z = 5 \\ x + 2y - 4z = 6 \\ 2x - 3y + z = -1 \end{cases}$.

This system forms the augmented matrix $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 1 & 2 & -4 & 2 \\ 2 & -3 & 1 & -1 \end{array}\right]$, which indicates $\mathrm{D} = -8$.
Meanwhile, $D_x = -64$, $D_y = -56$, and $D_z = -32$.

Therefore, the solution is $\begin{cases} x = \frac{-64}{-8} = 8 \\ y = \frac{-56}{-8} = 7 \\ z = \frac{-32}{-8} = 4 \end{cases}$.

Practice Questions (No CALC)

Part I. Calculate the determinants of following matrices:

*$\det(A) \times \det(B) = \det(AB)$, this will help you bit more than other properties of determinants can, but other properties can help you as well.

Matrix	Determinant	Matrix	Determinant
(1) $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$	7	(2) $\begin{bmatrix} 4 & 2 \\ 8 & 7 \end{bmatrix}$	12
(3) $\begin{bmatrix} 9 & 10 \\ 3 & 4 \end{bmatrix}$	6	(4) $\begin{bmatrix} 1 & 6 \\ -4 & 9 \end{bmatrix}$	33
(5) $\begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$	0	(6) $\begin{bmatrix} -5 & -4 \\ 1 & -7 \end{bmatrix}$	39
(7) $\begin{bmatrix} 10 & 0 \\ 5 & 20 \end{bmatrix}$	200	(8) $\begin{bmatrix} 12 & 12 \\ -20 & 8 \end{bmatrix}$	336
(9) $\begin{bmatrix} 42 & 49 \\ 35 & 77 \end{bmatrix}$	1519	(10) $\begin{bmatrix} 28 & 40 \\ -64 & 12 \end{bmatrix}$	2896
(11) $\begin{bmatrix} -132 & -99 \\ 121 & 55 \end{bmatrix}$	-4719	(12) $\begin{bmatrix} 13 & 15 \\ 0 & 0 \end{bmatrix}$	0
(13) $\begin{bmatrix} -4 & 2 \\ -10 & 1 \end{bmatrix}$	16	(14) $\begin{bmatrix} 300 & -15 \\ -15 & -105 \end{bmatrix}$	-31725
(15) $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 6 \\ 9 & 6 & 3 \end{bmatrix}$	0	(16) $\begin{bmatrix} 2 & 4 & 1 \\ 5 & 3 & 7 \\ 5 & 2 & 1 \end{bmatrix}$	93
(17) $\begin{bmatrix} 6 & 3 & 8 \\ 4 & 8 & 2 \\ 6 & 2 & 7 \end{bmatrix}$	-56	(18) $\begin{bmatrix} 1 & 5 & 3 \\ 2 & 3 & 1 \\ 4 & 3 & 5 \end{bmatrix}$	-36
(19) $\begin{bmatrix} 10 & -2 & 5 \\ -9 & 3 & 3 \\ 6 & 2 & -3 \end{bmatrix}$	-312	(20) $\begin{bmatrix} 18 & 9 & -3 \\ 0 & 6 & -12 \\ -12 & -6 & 0 \end{bmatrix}$	-216
(21) $\begin{bmatrix} -18 & 9 & 0 \\ 3 & 6 & 9 \\ 0 & 0 & 12 \end{bmatrix}$	-1620	(22) $\begin{bmatrix} 5 & -2 & 9 \\ -3 & 5 & -7 \\ 3 & 1 & 4 \end{bmatrix}$	-9
(23) $\begin{bmatrix} 64 & 16 & 12 \\ 56 & 0 & 40 \\ 36 & 20 & 28 \end{bmatrix}$	-39808	(24) $\begin{bmatrix} 75 & 15 & 90 \\ 105 & 135 & 15 \\ 30 & 45 & 0 \end{bmatrix}$	16875
(25) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	2	(26) $\begin{bmatrix} 145 & 156 & 167 \\ 0 & 0 & 0 \\ 287 & 276 & 265 \end{bmatrix}$	0
(27) $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 12 & 10 & 4 & 16 \\ 12 & 20 & 12 & 16 \\ 18 & 14 & 0 & 2 \end{bmatrix}$	-5824	(28) $\begin{bmatrix} 3 & 15 & 9 & 27 \\ 6 & 15 & 18 & 21 \\ 0 & 30 & 42 & 15 \\ 39 & 0 & 0 & 12 \end{bmatrix}$	98415
(29) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$	2	(30) $\begin{bmatrix} 10 & 10 & 10 & 20 \\ 10 & 10 & 20 & 20 \\ 10 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \end{bmatrix}$	12

Part II. Find the inverse matrix of following matrices with (a) Gauss-Jordan Method and (b) determinant and adjugate matrices. Later, (c) reflect on the efficiency of both methods on finding inverse matrices of different size matrices.

Matrix	Inverse	Matrix	Inverse

(1) $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$	$\begin{bmatrix} \frac{5}{7} & -\frac{3}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$	(2) $\begin{bmatrix} 4 & 2 \\ 8 & 7 \end{bmatrix}$	$\begin{bmatrix} \frac{7}{12} & -\frac{1}{6} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$
(3) $\begin{bmatrix} 9 & 10 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} \frac{2}{3} & -\frac{5}{3} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$	(4) $\begin{bmatrix} 1 & 6 \\ -4 & 9 \end{bmatrix}$	$\begin{bmatrix} \frac{3}{11} & -\frac{2}{11} \\ \frac{4}{33} & \frac{1}{33} \end{bmatrix}$
(5) $\begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$	*No inverse*	(6) $\begin{bmatrix} -5 & -4 \\ 1 & -7 \end{bmatrix}$	$\begin{bmatrix} -\frac{7}{39} & \frac{4}{39} \\ -\frac{1}{39} & -\frac{5}{39} \end{bmatrix}$
(7) $\begin{bmatrix} 10 & 0 \\ 5 & 20 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{10} & 0 \\ -\frac{1}{40} & \frac{1}{20} \end{bmatrix}$	(8) $\begin{bmatrix} 12 & 12 \\ -20 & 8 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{42} & -\frac{1}{28} \\ \frac{5}{84} & \frac{1}{28} \end{bmatrix}$
(9) $\begin{bmatrix} 42 & 49 \\ 35 & 77 \end{bmatrix}$	$\begin{bmatrix} \frac{11}{217} & -\frac{1}{31} \\ -\frac{5}{217} & \frac{6}{217} \end{bmatrix}$	(10) $\begin{bmatrix} 28 & 40 \\ -64 & 12 \end{bmatrix}$	$\begin{bmatrix} \frac{3}{724} & -\frac{5}{362} \\ \frac{4}{181} & \frac{7}{724} \end{bmatrix}$
(11) $\begin{bmatrix} -132 & -99 \\ 121 & 55 \end{bmatrix}$	$\begin{bmatrix} \frac{5}{429} & \frac{3}{143} \\ -\frac{1}{39} & -\frac{4}{143} \end{bmatrix}$	(12) $\begin{bmatrix} 13 & 15 \\ 0 & 0 \end{bmatrix}$	*No inverse*
(13) $\begin{bmatrix} -4 & 2 \\ -10 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{16} & -\frac{1}{8} \\ \frac{5}{8} & -\frac{1}{4} \end{bmatrix}$	(14) $\begin{bmatrix} 300 & -15 \\ -15 & -105 \end{bmatrix}$	$\frac{1}{2085}\begin{bmatrix} \frac{7}{2085} & \frac{1}{2085} \\ \frac{1}{2085} & \frac{4}{417} \end{bmatrix}$
(15) $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 6 \\ 9 & 6 & 3 \end{bmatrix}$	*No inverse*	(16) $\begin{bmatrix} 2 & 4 & 1 \\ 5 & 3 & 7 \\ 5 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} -\frac{11}{93} & -\frac{2}{93} & \frac{25}{93} \\ \frac{10}{31} & -\frac{1}{31} & -\frac{3}{31} \\ -\frac{5}{93} & \frac{16}{93} & -\frac{14}{93} \end{bmatrix}$
(17) $\begin{bmatrix} 6 & 3 & 8 \\ 4 & 8 & 2 \\ 6 & 2 & 7 \end{bmatrix}$	$\begin{bmatrix} -\frac{13}{14} & \frac{5}{56} & \frac{29}{28} \\ \frac{2}{7} & \frac{3}{28} & -\frac{5}{14} \\ \frac{5}{7} & -\frac{3}{28} & -\frac{9}{14} \end{bmatrix}$	(18) $\begin{bmatrix} 1 & 5 & 3 \\ 2 & 3 & 1 \\ 4 & 3 & 5 \end{bmatrix}$	$\begin{bmatrix} -\frac{3}{36} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{6} & \frac{7}{36} & -\frac{5}{36} \\ \frac{1}{6} & -\frac{17}{36} & \frac{7}{36} \end{bmatrix}$
(19) $\begin{bmatrix} 10 & -2 & 5 \\ -9 & 3 & 3 \\ 6 & 2 & -3 \end{bmatrix}$	$\begin{bmatrix} \frac{5}{104} & -\frac{1}{78} & \frac{7}{104} \\ \frac{3}{104} & \frac{5}{26} & \frac{25}{104} \\ \frac{3}{26} & \frac{4}{39} & -\frac{1}{26} \end{bmatrix}$	(20) $\begin{bmatrix} 18 & 9 & -3 \\ 0 & 6 & -12 \\ -12 & -6 & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{3} & -\frac{1}{12} & \frac{5}{12} \\ -\frac{2}{3} & \frac{1}{6} & -1 \\ -\frac{1}{3} & 0 & -\frac{1}{2} \end{bmatrix}$

(21) $\begin{bmatrix} -18 & 9 & 0 \\ 3 & 6 & 9 \\ 0 & 0 & 12 \end{bmatrix}$	$\begin{bmatrix} -\frac{2}{45} & \frac{1}{15} & -\frac{1}{20} \\ \frac{1}{45} & \frac{2}{15} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{12} \end{bmatrix}$	(22) $\begin{bmatrix} 5 & -2 & 9 \\ -3 & 5 & -7 \\ 3 & 1 & 4 \end{bmatrix}$	$\begin{bmatrix} -\frac{3}{5} & -\frac{1}{45} & \frac{59}{45} \\ \frac{1}{5} & \frac{7}{45} & -\frac{8}{45} \\ \frac{2}{5} & -\frac{1}{45} & -\frac{31}{45} \end{bmatrix}$
(23) $\begin{bmatrix} 64 & 16 & 12 \\ 56 & 0 & 40 \\ 36 & 20 & 28 \end{bmatrix}$	$\begin{bmatrix} \frac{25}{1244} & \frac{13}{2488} & -\frac{5}{311} \\ \frac{1}{311} & -\frac{85}{2488} & \frac{59}{1244} \\ -\frac{35}{1244} & \frac{11}{622} & \frac{7}{311} \end{bmatrix}$	(24) $\begin{bmatrix} 75 & 15 & 90 \\ 105 & 135 & 15 \\ 30 & 45 & 0 \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{25} & \frac{6}{25} & -\frac{53}{75} \\ \frac{2}{75} & -\frac{4}{25} & \frac{37}{75} \\ \frac{1}{25} & -\frac{13}{75} & \frac{38}{75} \end{bmatrix}$
(25) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$	(26) $\begin{bmatrix} 45 & 56 & 67 \\ 0 & 0 & 0 \\ 87 & 76 & 65 \end{bmatrix}$	$No\ inverse$
(27) $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 12 & 10 & 4 & 16 \\ 12 & 20 & 12 & 16 \\ 18 & 14 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} \frac{107}{364} & -\frac{9}{364} & -\frac{101}{728} & \frac{12}{91} \\ -\frac{33}{91} & \frac{3}{182} & \frac{16}{91} & -\frac{8}{91} \\ \frac{165}{364} & -\frac{53}{364} & -\frac{69}{728} & \frac{10}{91} \\ -\frac{3}{28} & \frac{3}{28} & \frac{1}{56} & -\frac{1}{14} \end{bmatrix}$	(28) $\begin{bmatrix} 3 & 15 & 9 & 27 \\ 6 & 15 & 18 & 21 \\ 0 & 30 & 42 & 15 \\ 39 & 0 & 0 & 12 \end{bmatrix}$	$\begin{bmatrix} \frac{8}{729} & -\frac{32}{729} & \frac{4}{243} & \frac{23}{729} \\ \frac{772}{3645} & -\frac{1387}{3645} & \frac{143}{1215} & \frac{154}{3645} \\ -\frac{101}{729} & \frac{161}{729} & -\frac{10}{243} & -\frac{17}{729} \\ -\frac{26}{729} & \frac{104}{729} & -\frac{13}{243} & -\frac{14}{729} \end{bmatrix}$
(29) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -\frac{1}{2} \end{bmatrix}$	(30) $\begin{bmatrix} 10 & 10 & 10 & 20 \\ 10 & 10 & 20 & 20 \\ 10 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -\frac{1}{10} & \frac{1}{10} \\ 0 & -\frac{1}{10} & \frac{1}{10} & 0 \\ -\frac{1}{10} & \frac{1}{10} & 0 & 0 \\ \frac{1}{10} & 0 & 0 & -\frac{1}{20} \end{bmatrix}$

Part III. Solve the following systems of equations with (a) augmented matrices, (b) inverse matrices, and (c) Cramer's Rule.

System	Solutions	System	Solutions
(1) $\begin{cases} eq1: 2x + y = 4 \\ eq2: 2x - y = 0 \end{cases}$	$\begin{cases} x = 1 \\ y = 2 \end{cases}$	(2) $\begin{cases} eq1: 3x + 4y = 13 \\ eq2: 6x + y = 19 \end{cases}$	$\begin{cases} x = 3 \\ y = 1 \end{cases}$
(3) $\begin{cases} eq1: x + 4 = 2y \\ eq2: x - 3y = -9 \end{cases}$	$\begin{cases} x = 6 \\ y = 5 \end{cases}$	(4) $\begin{cases} eq1: x + y = 10 \\ eq2: y - x = 4 \end{cases}$	$\begin{cases} x = 3 \\ y = 7 \end{cases}$
(5) $\begin{cases} eq1: 2x - y = 2 \\ eq2: x + y = 22 \end{cases}$	$\begin{cases} x = 8 \\ y = 14 \end{cases}$	(6) $\begin{cases} eq1: 5x + 9 = y \\ eq2: y - x = 65 \end{cases}$	$\begin{cases} x = 14 \\ y = 79 \end{cases}$
(7) $\begin{cases} eq1: x + y = 17 \\ eq2: 5x + 12y = 120 \end{cases}$	$\begin{cases} x = 12 \\ y = 5 \end{cases}$	(8) $\begin{cases} eq1: y - x = 27 \\ eq2: 0.2y - 0.5x = 0 \end{cases}$	$\begin{cases} x = 18 \\ y = 45 \end{cases}$
(9) $\begin{cases} eq1: x + 9 = y \\ eq2: x - 2y = -22 \end{cases}$	$\begin{cases} x = 4 \\ y = 13 \end{cases}$	(10) $\begin{cases} eq1: 3x - 2y = 0 \\ eq2: y - x = 0.5x \end{cases}$	$\begin{cases} x = 38 \\ y = 57 \end{cases}$
(11) $\begin{cases} eq1: 3x - y + 2z = 7 \\ eq2: 2x + 3y - z = 11 \\ eq3: x + y + z = 7 \end{cases}$	$\begin{cases} x = 2 \\ y = 3 \\ z = 2 \end{cases}$	(12) $\begin{cases} eq1: -x + y - z = 6 \\ eq2: x - 4y + 2z = -22 \\ eq3: 4x + 3y - 2z = -5 \end{cases}$	$\begin{cases} x = -4 \\ y = 7 \\ z = 5 \end{cases}$

(13) $\begin{cases} eq1: x-y+z=13 \\ eq2: 6x-y-z=10 \\ eq3: -9y-z=0 \end{cases}$	$\begin{cases} x=3 \\ y=-1 \\ z=9 \end{cases}$	(14) $\begin{cases} eq1: x-y+z=4 \\ eq2: 3x+y+z=0 \\ eq3: 2x+y+z=-3 \end{cases}$	$\begin{cases} x=3 \\ y=-4 \\ z=-5 \end{cases}$
(15) $\begin{cases} eq1: x-y-z=0 \\ eq2: x+y=-9 \\ eq3: x-z=-7 \end{cases}$	$\begin{cases} x=-2 \\ y=-7 \\ z=5 \end{cases}$	(16) $\begin{cases} eq1: x+y-z=1 \\ eq2: x-y+3z=49 \\ eq3: 2x+y+z=30 \end{cases}$	$\begin{cases} x=11 \\ y=-1 \\ z=9 \end{cases}$
(17) $\begin{cases} eq1: 2x+y+z=13 \\ eq2: 3x-y+3z=-1 \\ eq3: x+y-z=1 \end{cases}$	$\begin{cases} x=-4 \\ y=13 \\ z=8 \end{cases}$	(18) $\begin{cases} eq1: x+y+2z=0 \\ eq2: x-y-z=5 \\ eq3: 4x-8y-z=3 \end{cases}$	$\begin{cases} x=4 \\ y=2 \\ z=-3 \end{cases}$
(19) $\begin{cases} eq1: 2x-y+z=0 \\ eq2: 7x-y-z=-5 \\ eq3: x+y+2z=18 \end{cases}$	$\begin{cases} x=1 \\ y=7 \\ z=5 \end{cases}$	(20) $\begin{cases} eq1: 2x+y-z=7 \\ eq2: 4x-2y+z=9 \\ eq3: 2x-y+z=13 \end{cases}$	$\begin{cases} x=4 \\ y=8 \\ z=9 \end{cases}$