Chapter 11-3: Methods of Solving Roots for Polynomial Equations

Chapter Guidelines:

- Polynomial equations are equations where a polynomial's value is equal to zero. The solutions of this equation are called the roots of polynomial.
- Roots of polynomials can be algebraically attained by using formulas, factorization, and sometimes solving systems of equations.

1: Basic Method and Concept of Solving Roots

Factorizing a polynomial can help us inspect the polynomials in different ways, and one of its most significant purpose is to help solve the "roots" of a polynomial. **Roots** of a polynomial are specific values such that if the variable of polynomial is equivalent to the numerical value of the root, then the value of the polynomial expression equals 0.
For example, in the factored polynomial $(x-2)(x-3)$, the roots of this polynomial are 2 and 3, because the entire expression's value is 0 when $x=2$ or $x=3$.

To solve the roots of a polynomial, we can construct a **polynomial equation**: an equation that states "polynomial expression" $=0$.

A polynomial equation is an equation that state the equality: for a polynomial P, $P=0$.

By constructing a polynomial equation, we can solve for the variable in the polynomial to find the roots of it. This method is applicable because finding the solution of a polynomial equation is exactly solving for a value the variable equates to, given that the value of polynomial's expression is equivalent to 0.

For example: we can find the roots in polynomial $P(x)=x^2+2x+1$ by the following method:

Step 1: Construct a polynomial equation for this polynomial $P(x)$.
Step 2: Factor the polynomial, and the polynomial equation becomes $(x+1)^2=0$.
Step 3: For a product to equal 0, either the multiplicand or the multiplier has to be 0, or both has to be 0. In this case, $x+1=0$.
Step 4: Solve for all x in the equation, and $x=root$. $x=-1$.

In the above example, $P(x)$ has two roots and equation $P(x)=0$ has two solutions, and the two roots (solutions) both have a same value: -1. This repeated root is also known as a **multiple root**, and because the root -1 has been repeated twice, it is called a **double root**.

Or we can square root both sides of equation.
Knowing that the equation is $(x+1)^2 = 0$, squaring root both sides of equations would produce the equation $\sqrt{(x+1)^2} = \sqrt{0}$, which would show $(x+1) = 0$ an provide the solution $x = -1$.

The above method is best used when the equation is not equal to 0.
For example, a variation of the above equation would be $x^2 + 2x + 1 = 1$:

Step 1: Observe both sides of equations and think about what can be done to simplify it.
Step 2: Factor the polynomial at left hand side, and the equation becomes $(x+1)^2 = 1$.
Step 3: Square root both sides of equations, and equation $(x+1) = \pm 1$ is provided. $x = -1 \pm 1$.
Step 4: Solve for all x in the equation, and $x = root.\ x = 0\ or\ 2$.

In the following section we will discuss properties of polynomials with double roots.

2: Completing the Square

A single-variable second-degree polynomial with a double root can be factored as $(x+d)^2$, which can be foiled into polynomial $x^2 + 2dx + d^2$.

Using this above property, we can solve quadratic equations by method "completing the square".

Completing the square is a method of solving quadratic equations about making one side of the quadratic equation a polynomial with double root, such as $(x-d)^2$ where d is a double-root, then taking the square roots of both sides of quadratic equation to attain a solution.
Below is a description of the procedure of this method.

Step 1: Construct the equation you want to solve: $x^2 + 2dx + e = f$.
Step 2: Make one side of this quadratic equation a polynomial with double root. To do that, add $(d^2 - e)$ to both sides of equation: $x^2 + 2dx + e + (d^2 - e) = f + (d^2 - e) = x^2 + 2dx + d^2$.
Step 3: Take square root of both sides of equation. $(x+d)^2 = d^2 - e + f$ is the original equation.
The square-rooted equation is $(x+d) = \pm\sqrt{d^2 - e + f}$.
Step 4: Solve for x. From the above equation $(x+d) = \pm\sqrt{d^2 - e + f}$,
we attain $x = -d \pm \sqrt{d^2 - e + f}$.

Below is an example of using completing the square to attain the roots of $P(x) = x^2 + 4x + 2$:

Step 1: Construct a polynomial equation for this polynomial $P(x)$.
Step 2: The current equation is $x^2 + 4x + 2 = 0$, and we are trying to convert it into an equation that is $(x + r)^2 = x^2 + 2rx + r^2 = d$, where r and d are numbers. $2r = 4$, so $r^2 = 4$.
The current constant term on the left side of equation is 2, and to make the left side of equation a double-root polynomial, we add 2 on both sides of equations.
Now the equation becomes $x^2 + 4x + 4 = (x + 2)^2 = 0 + 2$.
Step 3: Square root both sides of equations, the equation becomes $x + 2 = \pm\sqrt{2}$.
Step 4: Solve for x from above equation. $x = -2 \pm \sqrt{2}$.

In the above example, the polynomial has two distinct roots.

3: Quadratic Formula

Quadratic Formula is a formula used to solve solutions of any single-variable second-degree polynomial equations, including polynomial equations for single-variable second-degree polynomials. In fact, it can only be used on this type of polynomial.

The derivation of quadratic formula is completed by using the aforementioned method "completing the square".
Below is the derivation of quadratic formula:

A polynomial equation for any single-variable second-degree polynomial has the format:
$ax^2 + bx + c = 0$, where a, b, c refers to the coefficients in the polynomial.
To find a solution we will complete the square: the equation becomes $a(x^2 + \frac{bx}{a} + \frac{b^2}{4a^2}) + c = \frac{b^2}{4a}$.
At the current step, the equation is $\frac{\frac{b^2}{4a} - c}{a} = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2-4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$.
Square root both sides of equations, and we will attain equation $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \frac{\pm\sqrt{b^2-4ac}}{2a}$.
Solve for x: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

For an equation $ax^2 + bx + c = 0$, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

From components of the quadratic formula, mathematicians developed a method to inspect how many roots does a single-variable second-degree polynomial has. In short, they developed the **discriminant of single-variable second-degree polynomials**, or a quality that classifies same type of expressions into different categories, in this case the amount of distinct roots a polynomial has.

For a polynomial $P(x) = ax^2 + bx + c$, the component of quadratic formula that determined whether multiple x exists or no x is real number is the component $\pm\sqrt{b^2 - 4ac}$. Therefore, the discriminant of a quadratic polynomial is $b^2 - 4ac$, and the analysis on this discriminant follows:

If the value of $\sqrt{b^2-4ac}$ is 0, or that $b^2-4ac=0$, then this polynomial has a double root $x = -\frac{b}{2a} \pm 0$.
Then, if $b^2-4ac>0$, then $\sqrt{b^2-4ac}$ is a real number, and there will be two distinct roots.
However, if $b^2-4ac<0$, then since $\sqrt{b^2-4ac}$ is not a real number, there will be two imaginary roots and zero real roots for polynomial $P(x)$.

In summary:

For polynomial $P(x)=ax^2+bx+c$, the discriminant of $P(x)$ is b^2-4ac.

$$\begin{cases} if\ b^2-4ac>0, P(x)\ has\ two\ distinct\ roots, roots\ are\ all\ real \\ if\ b^2-4ac=0, P(x)\ has\ one\ disctinct\ root\ (double\ root), roots\ are\ all\ real \\ if\ b^2-4ac<0, P(x)\ has\ no\ real\ root\ and\ two\ distinct\ imaginary\ roots. \end{cases}$$

4: Vieta Formula

Vieta Formula is a set of algebraic identities describing the relationship between a polynomial's coefficients and roots.

Below is an introduction to Vieta Formulas for single-variable second-degree polynomials:

For example: for a second-degree polynomial $P(x)$ that has two roots, which we will notate as r_1 and r_2, the factored form of this polynomial form should be $P(x)=(x-r_1)(x-r_2)$, and would be foiled out as $x^2-r_1x-r_2x+r_1r_2$, or $x^2-(r_1+r_2)x+r_1r_2$.
Comparing this foiled polynomial to the general equation for its type of polynomial, which is $P(x)=ax^2+bx+c$, we can see that the fraction $\frac{b}{a}$ is equivalent to the negative value of sum of roots, the fraction $\frac{c}{a}$ is equivalent to the product of roots.

In the above example, we figured the relationship between the coefficients and the roots of a single-variable second-degree polynomial, in other words the Vieta Formula for this type of polynomial. In the following analysis, we will apply the same procedures to figure the Vieta formulas for a single-variable cubic (third-degree) polynomial:

Find the Vieta Formula for a cubic polynomial Q(x).
$Q(x)=(x-r_1)(x-r_2)(x-r_3)$;
$Q(x)=(x^2-r_1x-r_2x+r_1r_2)(x-r_3)$;
$Q(x)=(x^3-r_1x^2-r_2x^2-r_3x^2+r_1r_2x+r_2r_3x+r_1r_3x-r_1r_2r_3)$.
This can be summarized as $Q(x)=x^3-(r_1+r_2+r_3)x^2+(r_1r_2+r_2r_3+r_1r_3)x-(r_1r_2r_3)$.
This gives us the algebraic shortcut to coefficients of $Q(x)=ax^3+bx^2+cx+d$ in terms of its roots: $\frac{b}{a}$ is negative value of sum of roots, $\frac{c}{a}$ is sum of products of all possible combinations of two individual roots, and $\frac{d}{a}$ is now negative value of product of all roots.

We can observe this trend and conclude that Vieta Formula will show the algebraic shortcut of n-degree polynomials in the format:

For a polynomial equation:
$P(x) = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \ldots + a_nx^1 + a_{n+1}x^0 = 0$, where $n \in N$ and a_k is the coefficient of k^{th} term counting from the highest-degree term, its Vieta formulas follows:

$$a_1 = a_1$$

$$\frac{a_2}{a_1} = (-1)^{(2-1)(r_1+r_2+\cdots+r_n)}$$

$$\frac{a_3}{a_1} = (-1)^{(3-1)(r_1r_2+r_1r_3+r_1r_4+\cdots+r_{n-1}r_n)}$$

$$(\ldots)$$

$$\frac{a_{n+1}}{a_1} = (a_1a_2a_3\ldots a_{n+1})$$

Since we have yet discussed solving a system of equations, we will explain the application of Vieta Formula in the next chapter. Just keep in mind that this property of polynomials is useful.

Practice Questions (No CALC)

Part I. Provide the number of distinct real roots using discriminant, then provide the roots of following polynomials using completing the square.

Polynomial Equation	Number of Distinct Real Roots	Answer
(1) $3x^2 + 4x + 1 = 0$	2	$x = 1 \text{ or } -\frac{1}{3}$
(2) $x^2 + 2x = 0$	2	$x = -2 \text{ or } 0$
(3) $4x^2 + 8x + 4 = 0$	1	$x = -1$
(4) $12x^2 + 24x - 11 = 0$	2	$x = \frac{-6 \pm \sqrt{39}}{6}$
(5) $3x^2 + 18x + 24 = 0$	2	$x = -4 \text{ or } -2$
(6) $20x^2 + 60x - 31 = 0$	2	$x = \frac{-15 \pm 2\sqrt{95}}{10}$
(7) $x^2 - 8x + 9 = 0$	2	$x = 4 \pm \sqrt{7}$
(8) $3x^2 - 19x - 11 = 0$	2	$x = \frac{19 \pm \sqrt{493}}{6}$
(9) $x^2 - 18x + 11 = 0$	2	$x = 9 \pm \sqrt{70}$
(10) $x^2 - 27x - 17 = 0$	2	$x = \frac{27 \pm \sqrt{797}}{6}$
(11) $16x^2 - 18x + 1 = 0$	2	$x = \frac{9 \pm \sqrt{65}}{6}$
(12) $2x^2 - 10x - 5 = 0$	2	$x = \frac{5 \pm \sqrt{35}}{6}$
(13) $130x^2 + 169x + 312 = 0$	0	N/A
(14) $\sqrt{2}x^2 + (2\sqrt{2} - 2)x + 1 = 0$	0	N/A
(15) $\sqrt{3}x^2 + (3\sqrt{3} - 1)x - 3 = 0$	2	$x = -3 \text{ or } \frac{\sqrt{3}}{3}$

Part II. Provide the roots of following polynomials using quadratic formula.

Polynomial Equation	Number of Distinct Real Roots	Answer
(1) $3x^2 + 15x + 1 = 0$	2	$x = \frac{-15 \pm \sqrt{213}}{6}$
(2) $2x^2 + 17x - 4 = 0$	2	$x = \frac{-17 \pm \sqrt{321}}{4}$
(3) $4x^2 - 7x - 3 = 0$	2	$x = \frac{7 \pm \sqrt{97}}{2}$
(4) $8x^2 - 9x + 2 = 0$	2	$x = \frac{-9 \pm \sqrt{17}}{16}$
(5) $x^2 + 3 = 0$	0	N/A
(6) $4x^2 - 2 = 0$	2	$x = \frac{\pm\sqrt{2}}{2}$

(7) $6x^2 + 13x - 2 = 0$	2	$x = \frac{-13 \pm \sqrt{217}}{12}$
(8) $17x^2 - 51x + 34 = 0$	2	$x = -2\ or - 1$
(9) $x^2 - 22x - 3 = 0$	2	$x = 11 \pm 2\sqrt{31}$
(10) $x^2 - 17x + 16 = 0$	2	$x = 16\ or\ 1$
(11) $-12x^2 - 72x - 108 = 0$	1	$x = -3$
(12) $x^2 + 11x + 23 = 0$	2	$x = \frac{-11 \pm \sqrt{29}}{2}$
(13) $9x^2 + 6x + 1 = 0$	1	$x = -\frac{1}{3}$
(14) $x^2 + 1001x + 1001 = 0$	2	$x = \frac{-1001 \pm \sqrt{997997}}{2}$
(15) $2x^2 + 1600x + 1600 = 0$	2	$x = -400 \pm 20\sqrt{398}$

Part III. Provide the roots of following polynomials in exact form using Vieta formula.

Polynomial Equation	Answer
(1) $x^2 + 5x - 6 = 0$	$x = -6\ or\ 1$
(2) $x^2 - 12x + 11 = 0$	$x = 1\ or\ 11$
(3) $16x^2 - 56x + 49 = 0$	$x = \frac{7}{4}$
(4) $3x^2 + 12x - 18 = 0$	$x = -2 \pm \sqrt{10}$
(5) $5x^2 - 13x + 8 = 0$	$x = 1\ or \frac{8}{5}$
(6) $7x^2 - 48x - 21 = 0$	$x = \frac{24 \pm \sqrt{723}}{7}$
(7) $11x^2 + 31x - 9 = 0$	$x = \frac{-31 \pm \sqrt{1357}}{22}$
(8) $4x^2 + 18x + 6 = 0$	$x = \frac{-9 \pm \sqrt{57}}{4}$
(9) $9x^2 + 21x + 6 = 0$	$x = -2 \text{ or} - \frac{1}{3}$
(10) $12x^2 + 108x + 243 = 0$	$x = -\frac{9}{2}$
(11) $16x^2 + 8x + 1 = 0$	$x = -\frac{1}{4}$
(12) $x^2 + 101x + 101 = 0$	$x = \frac{-101 \pm \sqrt{9797}}{2}$

Part IV. Solve the following polynomial equations:

(1) $8x^3 - 16x^2 - 2x + 4 = 0$	$x = -\frac{1}{2}\ or \frac{1}{2}\ or\ 2$
(2) $12x^4 - 72x^2 + 108 = 0$	$x = \pm\sqrt{3}$
(3) $13x^6 + 39x^3 + 26 = 0$	$x = -\sqrt[3]{2}\ or - 1$

(4) $(x-2)^2 = 8$	$x = 2 \pm 2\sqrt{2}$
(5) $(x^2 - 25x + 156.25)^2 = 14641$	$x = 1.5\ or\ 23.5$